

CENTRAL LIMIT THEOREM OF NONPARAMETRIC ESTIMATE OF SPECTRAL DENSITY FUNCTIONS OF SAMPLE COVARIANCE MATRICES

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ABSTRACT. A consistent kernel estimator of the limiting spectral distribution of general sample covariance matrices was introduced in Jing, Pan, Shao and Zhou (2010). The central limit theorem of the kernel estimator is proved in this paper.

1. INTRODUCTION

Spectral analysis of sample covariance matrices plays a very important role in multivariate statistical inference since many test statistics are defined by its eigenvalues or functionals. Let $\mathbf{X} = (X_{ij})_{p \times n}$ be independent and identically distributed (i.i.d.) real-valued random variables and \mathbf{T} be a $p \times p$ non-random Hermitian non-negative definite matrix with $(\mathbf{T}^{1/2})^2 = \mathbf{T}$. Define the sample covariance matrix by

$$\mathbf{A}_n = \frac{1}{n} \mathbf{T}^{1/2} \mathbf{X}_n \mathbf{X}_n^T \mathbf{T}^{1/2}$$

and its empirical spectral distribution $F_n^{\mathbf{A}}$ by

$$F_n^{\mathbf{A}}(x) = \frac{1}{p} \sum_{k=1}^p I(\lambda_k \leq x),$$

where $\lambda_k, k = 1, \dots, p$ denote the eigenvalues of \mathbf{A}_n . Instead of \mathbf{A}_n we also consider

$$\mathbf{B}_n = \frac{1}{n} \mathbf{X}_n^T \mathbf{T} \mathbf{X}_n,$$

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because the eigenvalues of \mathbf{A}_n and \mathbf{B}_n differ by $|n-p|$ zero eigenvalues. Suppose the ratio of the dimension and sample size $c_n = p/n$ tends to a positive constant c as $n \rightarrow \infty$. When $F^{\mathbf{T}}$ converges weakly to a distribution H , it is proved in Marcenko and Pastur [8], Yin [16] and Silverstein [12] that, with probability one, $F^{\mathbf{B}_n}(x)$ converges in distribution to an MP type distribution function $\underline{F}^{c,H}(x)$ whose Stieltjes transform $\underline{m}(z) = m_{\underline{F}^{c,H}}(z)$ is, for each $z \in \mathcal{C}^+ = \{z \in \mathcal{C} : \Im z > 0\}$, the unique solution to the equation

$$(1.1) \quad \underline{m} = - \left(z - c \int \frac{tdH(t)}{1 + t\underline{m}} \right)^{-1}.$$

Here the Stieltjes transform $m_F(z)$ for any probability distribution function $F(x)$ is given by

$$(1.2) \quad m_F(z) = \int \frac{1}{x - z} dF(x), \quad z \in \mathcal{C}^+.$$

Note that from (1.1) $\underline{m}(z)$ has an inverse

$$(1.3) \quad z = -\frac{1}{\underline{m}} + c \int \frac{t}{1 + t\underline{m}} dH(t).$$

Bai and Silverstein [2] established a far reaching central limit theorem (CLT) for the eigenvalues of \mathbf{A}_n , which makes possible the hypothesis testing of linear spectral statistics of sample covariance matrices indexed by analytic functions. Pan and Zhou [9] relaxed some restriction on the fourth moment of the underlying random variables. Lytova and Marcenko [7] and Bai, Wang and Zhou [3] further, respectively, extended Bai and Silverstein's theorem from the analytic test function to the one having fourth derivative when \mathbf{T} is the identity matrix. However, the limiting spectral distribution $\underline{F}^{c,H}$ is usually unknown for general T . It is also not clear if there is any CLT about $(F^{\mathbf{B}_n}(x) - \underline{F}^{c,H}(x))$, equivalently $(F_n^{\mathbf{A}_n}(x) - F^{c,H}(x))$ even in the normal population, here $F^{c,H}(x)$ is the limiting distribution of $F_n^{\mathbf{A}_n}(x)$. How can one make inference for $f_{c,H}(x)$ or $F^{c,H}(x)$ based on $F_n^{\mathbf{A}_n}(x)$ or $F^{\mathbf{B}_n}(x)$ without establishing CLTs?

Motivated by the “smoothing” ideas, Jing, Pan, Shao and Zhou [6] proposed the following kernel estimator of the density function of $\underline{F}^{c,H}(x)$ as

$$(1.4) \quad f_n(x) = \frac{1}{ph} \sum_{i=1}^p K\left(\frac{x - \lambda_i}{h}\right) = \frac{1}{h} \int K\left(\frac{x - y}{h}\right) dF_n^{\mathbf{A}_n}(y),$$

where h is the bandwidth. It was proved that $f_n(x)$ is a consistent estimator of $f_{c,H}(x)$ under some regularity conditions.

The main aim of this paper is to establish a CLT for $f_n(x)$. This provides an approach to making inference on the MP type distribution functions. To this end, we first list some technical conditions on the kernel function.

Suppose that the kernel function $K(x)$ satisfies

$$(1.5) \quad \lim_{|x| \rightarrow \infty} |xK(x)| = \lim_{|x| \rightarrow \infty} |xK'(x)| = 0,$$

$$(1.6) \quad \int K(x)dx = 1, \quad \int |xK'(x)|dx < \infty, \quad \int |K''(x)|dx < \infty.$$

and

$$(1.7) \quad \int xK(x)dx = 0, \quad \int x^2|K(x)|dx < \infty.$$

Let $z = u + iv$ with v being in a bounded interval, say $[-v_0, v_0]$ with $v_0 > 0$. Suppose that

$$(1.8) \quad \int_{-\infty}^{+\infty} |K^{(j)}(z)|du < \infty, \quad j = 0, 1, 2,$$

uniformly in $v \in [-v_0, v_0]$, where $K^{(j)}(z)$ denotes the k -th derivative of $K(z)$.

Some assumptions on $H_n(t) := F^{\mathbf{T}}$, are also needed. Introduce the interval

$$(1.9) \quad [\lambda_{\min}(\mathbf{T})(1 - \sqrt{c_n})^2, \lambda_{\max}(\mathbf{T})(1 + \sqrt{c_n})^2].$$

Denote the right and left end points of the above interval, respectively, by a_1 and a_2 . We then introduce a contour \mathcal{C}_1 as the union of four segments $\gamma_j, j = 1, 2, 3, 4$. Here

$$\gamma_1 = u - iv_0h, u \in [a_l, a_r], \quad \gamma_2 = u + iv_0h, u \in [a_l, a_r],$$

$$\gamma_3 = a_l + iv, v \in [-v_0h, v_0h], \quad \gamma_4 = a_r + iv, v \in [-v_0h, v_0h],$$

where a_l is any positive value smaller than the left end point of (1.9), a_r any value larger than the right end point of (1.9), and v_0 is specified in (1.8). Assume that on the contour \mathcal{C}_1

$$(1.10) \quad \left| 1 - c_n(\underline{m}_n^0(z))^2 \int \frac{t^2 dH_n(t)}{(1 + t\underline{m}_n^0(z))^2} \right| \geq M_1 \sqrt{v},$$

where $\Im(z) = v > 0$, M_1 is a positive constant and $\underline{m}_n^0(z)$ is the Stieltjes transform of the distribution function $\underline{F}^{c_n, H_n}(x)$ which is obtained from $\underline{F}^{c, H}(x)$ with c and H replaced by c_n and H_n . Also, on the contour \mathcal{C}_1 we assume that

$$(1.11) \quad \int \frac{dH_n(t)}{|1 + tE\underline{m}_n(z)|^4} < M$$

and that

$$(1.12) \quad \int \frac{dH_n(t)}{|1 + t\underline{m}_n^0(z)|^4} < M,$$

where $E\underline{m}_n(z)$ is the expectation of the Stieltjes transform of $F^{\mathbf{B}_n}$ and M is a constant independent of n and z . The main results are stated below.

Theorem 1. *Suppose that*

- 1) $h = h(n)$ is a sequence of positive constants satisfying
 - (1.13) $\lim_{n \rightarrow \infty} nh^{5/2} = \infty, \lim_{n \rightarrow \infty} nh^3 = 0, \lim_{n \rightarrow \infty} h = 0;$
 - 2) $K(x)$ satisfies (1.5)-(1.8) and is analytic on open interval including
- $$\left[\frac{a_2 - a_1}{h}, \frac{a_1 - a_2}{h} \right];$$
- 3) X_{ij} are i.i.d. with $EX_{11} = 0, \text{Var}(X_{11}) = 1, EX_{11}^4 = 3$ and $EX_{11}^{16} < \infty, c_n \rightarrow c \in (0, 1);$
 - 4) \mathbf{T} is a $p \times p$ non-random symmetric positive definite matrix with spectral norm bounded above by a positive constant such that $H_n = F^{\mathbf{T}}$ converges weakly to a distribution H . Also, H_n satisfies conditions (1.10)-(1.12);
 - 5) $F^{c_n, H}(x)$ has a compact support $[a, b]$ with $a > 0;$
 - 6) the function $K(x)$ and h satisfy

$$(1.14) \quad nh^2 \left[\int_{\frac{x-a}{h}}^{+\infty} yK(y)dy + \int_{-\infty}^{\frac{x-b}{h}} yK(y)dy \right] \rightarrow 0, \quad \int y^2 K(y) f_{c_n, H_n}''(y_0) dy < \infty$$

and

$$(1.15) \quad nh \left[1 - \int_{\frac{x-b}{h}}^{\frac{x-a}{h}} K(y) dy \right] \rightarrow 0,$$

where $f_{c_n, H_n}(x)$ is the density function of $F^{c_n, H_n}(x)$ and $y_0 = t(x - yh) + (1 - t)x$ with $t \in (0, 1)$ and $x \in (a, b)$.

Then, as $n \rightarrow \infty$, the limiting finite dimensional distributions of the processes of

$$(1.16) \quad nh \left(f_n(x) - f_{c_n, H_n}(x) \right), \quad x \in (a, b)$$

are multivariate normal with mean zero and covariance matrix $\sigma^2 I$, where

$$\sigma^2 = -\frac{1}{2\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K'(u_1) K'(u_2) \ln(u_1 - u_2)^2 du_1 du_2.$$

Remark 1. When \mathbf{T} is the identity matrix, (1.10) is true, which will be verified in Appendix 2, and conditions (1.11) and (1.12) also hold, see, (6.30) in [6] and (7.24). For general \mathbf{T} , (1.11) may be removed at the cost of higher moment of $H_n(t)$ and of a more stringent bandwidth. See Lemma 5 in Appendix 2.

Remark 2. It is easy to check that the Gaussian kernel function satisfies conditions specified in Theorem 1.

Theorem 1 is actually a corollary of the following theorem.

Theorem 2. *When the conditions (1.14), (1.15) and $\lim_{n \rightarrow \infty} nh^3 = 0$ in Theorem 1 are removed with the remaining conditions unchanged, Theorem 1 holds as well if the processes (1.16) are replaced by the processes*

$$nh \left[f_n(x) - \frac{1}{h} \int_a^b K\left(\frac{x-y}{h}\right) dF^{c_n, H_n}(y) \right].$$

We evaluate the quality of the estimate $f_n(x)$ by the mean integrated square error

$$\begin{aligned} L &= E \left(\int_a^b (f_n(x) - f_{c_n, H_n}(x))^2 dx \right) \\ &= \int_a^b \left(\text{Bias}(f_n(x)) \right)^2 dx + \int_a^b \text{Var}(f_n(x)) dx, \end{aligned}$$

where $\text{Bias}(f_n(x)) = Ef_n(x) - f_{c_n, H_n}(x)$. It is easy to verify that (see [14] and [10])

$$\frac{1}{h} \int K\left(\frac{x-y}{h}\right) dF^{c_n, H_n}(y) - f_{c_n, H_n}(x) = \frac{1}{2} h^2 (f^{c, H}(x))'' \int x^2 K(x) dx + O(h^3).$$

Although it is not rigorous from Theorem 2 we roughly have

$$Ef_n(x) - \frac{1}{h} \int K\left(\frac{x-y}{h}\right) dF^{c_n, H_n}(y) = o\left(\frac{1}{nh}\right)$$

and

$$\text{Var}(f_n(x)) = \frac{\sigma^2}{n^2 h^2} + o\left(\frac{\sigma^2}{n^2 h^2}\right).$$

These gives

$$L = \left(\frac{1}{2} h^2 (f_{c_n, H_n}(x))'' \int x^2 K(x) dx + O(h^3) + o\left(\frac{1}{nh}\right) \right)^2 + \frac{\sigma^2(b-a)}{n^2 h^2} + o\left(\frac{\sigma^2}{n^2 h^2}\right).$$

Differentiating the above with respect to h and setting it equal to zero, we see that the asymptotic optimal bandwidth is

$$(1.17) \quad h_* = \left(\frac{\sigma^2(b-a)}{2n^2 c_1^2} \right)^{1/6},$$

where $c_1 = \frac{1}{2} (f_{c_n, H_n}(x))'' \int x^2 K(x) dx < \infty$. This is different from the asymptotic optimal bandwidth $O(1/n^{1/5})$ in classical density estimates (see [14]).

As for $F_n(x) = \int_0^x f_n(y) dy$, we have the following result.

Theorem 3. *In addition to assumptions 2), 3), 4) and 5) in Theorem 1, suppose that*

$$\lim_{n \rightarrow \infty} nh^3 \sqrt{\ln \frac{1}{h}} \rightarrow \infty, \quad \lim_{n \rightarrow \infty} h \rightarrow 0.$$

Then, as $n \rightarrow \infty$, the limiting finite dimensional distributions of the processes of

$$(1.18) \quad \frac{n}{\sqrt{\ln \frac{1}{h}}} \left(F_n(x) - \int_{-\infty}^x \left[\frac{1}{h} \int K\left(\frac{t-y}{h}\right) dF^{c_n, H_n}(y) \right] dt \right)$$

are multivariate normal with mean zero and covariance matrix $\frac{1}{2\pi^2} I$.

Remark 3. We conjecture that Theorem 3 is still true if we substitute $F^{c_n, H_n}(x)$ for $\int_{-\infty}^x \left[\frac{1}{h} \int K\left(\frac{t-y}{h}\right) dF^{c_n, H_n}(y) \right] dt$. The convergence rate $n/\sqrt{\ln \frac{1}{h}}$ is consistent with the conjectured convergence rate $n/\sqrt{\log n}$ of the empirical spectral distributions of sample covariance matrices to the MP type distribution.

The paper is organized as follows. Theorem 2 is proved in Section 2 and Section 3. In Section 4 we present the proof of Theorem 3. Some technical lemmas are given in Appendix 1. Appendix 2 deals with Remark 1 and Theorem 1 and Appendix 3 gives the derivation of the variances and means in Theorem 2 and Theorem 3.

2. FINITE DIMENSIONAL CONVERGENCE OF THE PROCESSES

Throughout the paper, to save notation, M may stand for different constants on different occasions. This and the subsequent sections deal with Theorem 2 and the argument for handling $nh \left(\frac{1}{h} \int_a^b K\left(\frac{x-y}{h}\right) dF^{c_n, H_n}(y) - f_{c_n, H_n}(x) \right)$ is given at the end of Appendix 2.

Following the truncation steps in [2] we may truncate and re-normalize the random variables as follows

$$(2.1) \quad |X_{ij}| \leq \tau_n n^{1/2}, \quad EX_{ij}=0, \quad EX_{ij}^2 = 1,$$

where $\tau_n n^{1/3} \rightarrow \infty$. Based on this one may then verify that

$$(2.2) \quad EX_{11}^4 = 3 + O\left(\frac{1}{n}\right).$$

For any finite constants l_1, \dots, l_r and $x_1, \dots, x_r \in [a, b]$, by Cauchy's formula

$$(2.3) \quad \begin{aligned} & nh \left(\sum_{j=1}^r l_j (f_n(x_j) - \frac{1}{h} \int K\left(\frac{x_j-y}{h}\right) dF^{c_n, H_n}(y)) \right) \\ &= -\frac{1}{2\pi i} \oint_{c_1} \sum_{j=1}^r l_j K\left(\frac{x_j-z}{h}\right) X_n(z) dz, \end{aligned}$$

where $X_n(z) = \text{tr}(\mathbf{A}_n - z\mathbf{I})^{-1} - nm_{F^{c_n, H_n}}(z)$ and \mathcal{C}_1 is defined in the introduction.

From Fubini's theorem and (1.8) we obtain for $j = 0, 1, 2$.

$$\int_{a_l}^{a_r} \left[\frac{1}{h} \int_0^{v_0} |K^{(j)}(\frac{x-u}{h} + iv)| dv \right] du = \int_0^{v_0} \left[\frac{1}{h} \int_{a_l}^{a_r} |K^{(j)}(\frac{x-u}{h} + iv)| du \right] dv < \infty.$$

This implies for $u \in [a_l, a_r]$

$$(2.4) \quad \frac{1}{h} \int_0^{v_0} |K^{(j)}(\frac{x-u}{h} + iv)| dv < \infty, \quad j = 0, 1, 2.$$

For the sake of simplicity, write $\mathbf{A} = \mathbf{A}_n$. We now introduce some notation. Define $\mathbf{A}(z) = \mathbf{A} - z\mathbf{I}$, $\mathbf{A}_k(z) = \mathbf{A}(z) - \mathbf{s}_k \mathbf{s}_k^T$, and $\mathbf{s}_k = \mathbf{T}^{1/2} \mathbf{x}_k$ with \mathbf{x}_k being the k th column of \mathbf{X}_n . Let $E_k = E(\cdot | \mathbf{s}_1, \dots, \mathbf{s}_k)$ and E_0 denote the expectation. Set

$$\beta_k(z) = \frac{1}{1 + \mathbf{s}_k^T \mathbf{A}_k^{-1}(z) \mathbf{s}_k}, \quad \eta_k(z) = \mathbf{s}_k^T \mathbf{A}_k^{-1}(z) \mathbf{s}_k - \frac{1}{n} \text{tr} \mathbf{T} \mathbf{A}_k^{-1}(z),$$

$$b_1(z) = \frac{1}{1 + E \text{tr} \mathbf{T} \mathbf{A}_1^{-1}(z) / n}, \quad \beta_k^{tr}(z) = \frac{1}{1 + \text{tr} \mathbf{T} \mathbf{A}_k^{-1}(z) / n}.$$

We frequently use the following equalities:

$$(2.5) \quad \mathbf{A}^{-1}(z) - \mathbf{A}_k^{-1}(z) = -\beta_k(z) \mathbf{A}_k^{-1}(z) \mathbf{s}_k \mathbf{s}_k^T \mathbf{A}_k^{-1}(z);$$

$$(2.6) \quad \beta_1 = b_1 - b_1 \beta_1 \xi_1(z) = b_1 - b_1^2 \xi_1(z) + b_1^2 \beta_1 \xi_1^2(z)$$

where $\xi_1(z) = \mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{s}_1 - E n^{-1} \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{T}$. At this moment, we would point out that the length of the vertical lines of the contour of integral in (2.3) converges to zero. As a consequence, except $|b_1(z)|$ we can not expect $|\beta_k(z)|$ and $|\beta_k^{tr}(z)|$ to be bounded above by constants although they are bounded by $|z|/|v|$ (see [1]) (of course $v \neq 0$ in the cases of interest). Instead, the moments of $\beta_k(z)$ and $\beta_k^{tr}(z)$ are proved to be bounded. We summarize such estimates in Lemma 3 in Appendix 1. Sometimes we deal with the term $\frac{1}{n} \text{tr} \mathbf{T} \mathbf{A}_1^{-1}(z) \mathbf{T} \mathbf{A}_1^{-1}(\bar{z})$ in the following way: One may verify that

$$\text{Im}(1 + \frac{1}{n} \text{tr} \mathbf{T} \mathbf{A}_k^{-1}(z)) \geq v \lambda_{\min}(\mathbf{T}) \frac{1}{n} \text{tr} \mathbf{A}_k^{-1}(z) \mathbf{A}_k^{-1}(\bar{z}),$$

which implies that

$$(2.7) \quad |\beta_k^{tr}(z) \frac{1}{n} \text{tr} \mathbf{T} \mathbf{A}_k^{-1}(z) \mathbf{T} \mathbf{A}_k^{-1}(\bar{z})| \leq \frac{M}{|v|}.$$

We also frequently use the fact that $\|\mathbf{A}_k^{-1}(z)\| \leq 1/|v|$.

Write

$$\begin{aligned}
tr \mathbf{A}^{-1}(z) - E tr \mathbf{A}^{-1}(z) &= \sum_{k=1}^n \left(E_k tr \mathbf{A}^{-1}(z) - E_{k-1} tr \mathbf{A}^{-1}(z) \right) \\
&= \sum_{k=1}^n \left(E_k - E_{k-1} \right) tr \left[\mathbf{A}^{-1}(z) - \mathbf{A}_k^{-1}(z) \right] = - \sum_{k=1}^n \left(E_k - E_{k-1} \right) \left[\beta_k(z) \mathbf{s}_k^T \mathbf{A}_k^{-2}(z) \mathbf{s}_k \right] \\
(2.8) \quad &= - \sum_{k=1}^n \left(E_k - E_{k-1} \right) \left[\log \beta_k(z) \right]',
\end{aligned}$$

where in the third step one uses (2.5) and the derivative in the last equality is with respect to z . We then obtain from integration by parts that

$$\begin{aligned}
(2.9) \quad &\frac{1}{2\pi i} \oint K\left(\frac{x-z}{h}\right) (tr \mathbf{A}^{-1}(z) - E tr \mathbf{A}^{-1}(z)) dz \\
&= -\frac{1}{2\pi i} \sum_{k=1}^n (E_k - E_{k-1}) \oint K\left(\frac{x-z}{h}\right) \left[\log \beta_k(z) \right]' dz
\end{aligned}$$

$$\begin{aligned}
(2.10) \quad &= \frac{1}{h} \frac{1}{2\pi i} \sum_{k=1}^n (E_k - E_{k-1}) \oint K'\left(\frac{x-z}{h}\right) \log \left(\frac{\beta_k^{tr}(z)}{\beta_k(z)} \right) dz \\
&= \frac{1}{h} \frac{1}{2\pi i} \sum_{k=1}^n (E_k - E_{k-1}) \oint K'\left(\frac{x-z}{h}\right) \log \left(1 + \beta_k^{tr}(z) \eta_k(z) \right) dz
\end{aligned}$$

$$\begin{aligned}
(2.11) \quad &= \frac{1}{h} \frac{1}{2\pi i} \sum_{k=1}^n (E_k - E_{k-1}) \oint K'\left(\frac{x-z}{h}\right) \left(\beta_k^{tr}(z) \eta_k(z) + e_k(z) \right) dz
\end{aligned}$$

where

$$e_k(z) = \log(1 + \beta_k^{tr}(z) \eta_k(z)) - \beta_k^{tr}(z) \eta_k(z).$$

Below, consider $z \in \gamma_2$, the top horizontal line of the contour, unless it is further specified. We remind readers that $v = v_0 h$ on γ_2 . The next aim is to prove that

$$(2.12) \quad \frac{1}{h} \sum_{k=1}^n (E_k - E_{k-1}) \int K'\left(\frac{x-z}{h}\right) e_k(z) dz \xrightarrow{i.p.} 0.$$

By Lemma 4, we have for $m = 2, 4, 6$

$$(2.13) \quad E \left(|\eta_k(z)|^m |\mathbf{A}_k^{-1}(z)| \right) \leq \frac{M}{n^{m/2}} \left[\frac{1}{n} tr \mathbf{T} \mathbf{A}_k^{-1}(z) \mathbf{T} \mathbf{A}_k^{-1}(\bar{z}) \right]^{m/2}.$$

This, together with Lemma 3 in Appendix 1 and (2.7), gives

$$(2.14) \quad E |\beta_k^{tr}(z) \eta_k(z)|^4 = E \left(|\beta_k^{tr}(z)|^4 E(|\eta_k(z)|^4 |\mathbf{A}_k^{-1}(z)|) \right) \leq \frac{M}{n^2 v^2}.$$

It follows that

$$\sum_{k=1}^n P(|\beta_k^{tr}(z)\eta_k(z)| \geq 1/2) \leq 2^4 \sum_{k=1}^n E|\beta_k^{tr}(z)\eta_k(z)|^4 \leq \frac{M}{nv^2}.$$

Via (1.8), (2.14) and the inequality

$$(2.15) \quad |\log(1+x) - x| \leq M|x|^2, \text{ for } |x| \leq 1/2,$$

we obtain

$$\begin{aligned} (2.16) \quad & \frac{1}{h^2} E \left| \sum_{k=1}^n (E_k - E_{k-1}) \int K'(\frac{x-z}{h}) e_k(z) I(|\beta_k^{tr}(z)\eta_k(z)| < 1/2) du \right|^2 \\ & \leq \frac{M}{h^2} \sum_{k=1}^n E \left| \int K'(\frac{x-z}{h}) e_k(z) I(|\beta_k^{tr}(z)\eta_k(z)| < 1/2) du \right|^2 \\ & \leq \frac{M}{h^2} \sum_{k=1}^n \left[\int \int |K'(\frac{x-z_1}{h}) K'(\frac{x-z_2}{h})| \left(E|(\beta_k^{tr}(z_1)\eta_k(z_1))|^4 \right. \right. \\ & \quad \left. \left. \times E|(\beta_k^{tr}(z_2)\eta_k(z_2))|^4 \right)^{1/2} du_1 du_2 \right] \leq \frac{M}{nv^2}. \end{aligned}$$

Thus, (2.12) is proven. Therefore on γ_2

$$(2.17) \quad (2.9) = \frac{1}{2\pi i} \sum_{k=1}^n Y_k(x) + o_p(1),$$

where

$$Y_k(x) = E_k \left[\frac{1}{h} \int K'(\frac{x-z}{h}) (\beta_k^{tr}(z)\eta_k(z)) dz \right].$$

Apparently, $Y_k(z)$ is a martingale difference so that we may resort to the CLT for martingale (see Theorem 35.12 in [4]). As in (2.16), by (1.8) and (2.14) we have

$$\sum_{k=1}^n E|Y_k(z)|^4 \leq \frac{M}{nv^2}.$$

which ensures the Lyapunov condition for the CLT is satisfied.

Thus, it is sufficient to investigate the limit of the following covariance function

$$\begin{aligned} (2.18) \quad & -\frac{1}{4\pi^2} \sum_{k=1}^n E_{k-1}[Y_k(x_1)Y_k(x_2)] \\ & = -\frac{1}{4h^2\pi^2} \int \int K'(\frac{x_1-z_1}{h}) K'(\frac{x_2-z_2}{h}) \mathcal{C}_{n1}(z_1, z_2) dz_1 dz_2, \end{aligned}$$

where

$$\mathcal{C}_{n1}(z_1, z_2) = \sum_{k=1}^n E_{k-1} \left[E_k \left(\beta_k^{tr}(z_1) \eta_k(z_1) \right) E_k \left(\beta_k^{tr}(z_2) \eta_k(z_2) \right) \right].$$

By (2.7), (2.13) and (5.5)

$$\begin{aligned} & E \left(|(\beta_k^{tr}(z) - b_1(z)) \eta_k(z)|^2 \left| \mathbf{A}_k^{-1}(z) \right| \right) \\ & \leq \frac{M}{n^3} E \left(|\beta_k^{tr}(z) b_1(z) (tr \mathbf{A}^{-1}(z) \mathbf{T} - E tr \mathbf{A}^{-1}(z) \mathbf{T})|^2 \right. \\ & \quad \left. \times \frac{1}{n} tr \mathbf{T} \mathbf{A}_k^{-1}(z) \mathbf{T} \mathbf{A}_k^{-1}(\bar{z}) \left| \mathbf{A}_k^{-1}(z) \right| \right) \\ (2.19) \quad & \leq \frac{M}{n^3 v} |\beta_k^{tr}(z)| |tr \mathbf{A}^{-1}(z) \mathbf{T} - E tr \mathbf{A}^{-1}(z) \mathbf{T}|^2. \end{aligned}$$

This and Lemma 3 lead to

$$\begin{aligned} & E \left| (\beta_k^{tr}(z_1) - b_1(z_1)) \eta_k(z_1) E_k \left((\beta_k^{tr}(z_2) - b_1(z_2)) \eta_k(z_2) \right) \right| \\ & \leq \left[E |(\beta_k^{tr}(z_1) - b_1(z_1)) \eta_k(z_1)|^2 E |(\beta_k^{tr}(z_2) - b_1(z_2)) \eta_k(z_2)|^2 \right]^{1/2} \leq \frac{M}{n^3 v^4} \end{aligned}$$

and

$$E \left| (\beta_k^{tr}(z_1) - b_1(z_1)) \eta_k(z_1) E_k(\eta_k(z_2)) \right| \leq \frac{M}{n^2 v^{5/2}}.$$

It follows that

$$(2.20) \quad E \left| \mathcal{C}_n(z_1, z_2) - b_1(z_1) b_1(z_2) \sum_{k=1}^n E_{k-1} \left(E_k \eta_k(z_1) E_k \eta_k(z_2) \right) \right| \leq \frac{M}{n v^{5/2}}.$$

Note that for any non-random matrices \mathbf{B} and \mathbf{C}

$$\begin{aligned} & E(\mathbf{s}_1^T \mathbf{C} \mathbf{s}_1 - tr \mathbf{C})(\mathbf{s}_1^T \mathbf{B} \mathbf{s}_1 - tr \mathbf{B}) \\ & = n^{-2} (EX_{11}^4 - |EX_{11}^2|^2 - 2) \sum_{i=1}^p (\mathbf{T}^{1/2} \mathbf{C} \mathbf{T}^{1/2})_{ii} (\mathbf{T}^{1/2} \mathbf{B} \mathbf{T}^{1/2})_{ii} \\ (2.21) \quad & + |EX_{11}^2|^2 n^{-2} tr \mathbf{T}^{1/2} \mathbf{C} \mathbf{T} \mathbf{B}^T \mathbf{T}^{1/2} + n^{-2} tr \mathbf{T}^{1/2} \mathbf{C} \mathbf{T} \mathbf{B} \mathbf{T}^{1/2}. \end{aligned}$$

This implies that

$$\begin{aligned} & b_1(z_1) b_1(z_2) \sum_{k=1}^n E_{k-1} \left(E_k \eta_k(z_1) E_k \eta_k(z_2) \right) \\ (2.22) \quad & = (EX_{11}^4 - 3) b_1(z_1) b_1(z_2) \mathcal{C}_{n1}(z_1, z_2) + 2 b_1(z_1) b_1(z_2) \mathcal{C}_{n2}(z_1, z_2) \end{aligned}$$

$$(2.23) \quad = 2 b_1(z_1) b_1(z_2) \mathcal{C}_{n2}(z_1, z_2) + O\left(\frac{1}{n v^2}\right),$$

where

$$\begin{aligned}\mathcal{C}_{n1}(z_1, z_2) &= \frac{1}{n^2} \sum_{k=1}^n \sum_{j=1}^p (E_k(\mathbf{T}^{1/2} \mathbf{A}_k^{-1}(z_1) \mathbf{T}^{1/2})_{jj} E_k(\mathbf{T}^{1/2} \mathbf{A}_k^{-1}(z_2) \mathbf{T}^{1/2})_{jj}, \\ \mathcal{C}_{n2}(z_1, z_2) &= \frac{1}{n^2} \sum_{k=1}^n \text{tr} \mathbf{T}^{1/2} E_k(\mathbf{A}_k^{-1}(z_1)) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \mathbf{T}^{1/2},\end{aligned}$$

and in the last step one uses (2.2) and the fact that $|(E_k(\mathbf{T}^{1/2} \mathbf{A}_k^{-1}(z_1) \mathbf{T}^{1/2})_{jj}| \leq \frac{1}{v}$.

The next aim is to convert the random variables involved in $\mathcal{C}_{n2}(z_1, z_2)$ to their corresponding expectations. To this end, we introduce more notation and estimates, and establish a lemma. Define

$$\begin{aligned}\mathbf{A}_{kj}(z) &= \mathbf{A}(z) - \mathbf{s}_k \mathbf{s}_k^T - \mathbf{s}_j \mathbf{s}_j^T, \beta_{kj}(z) = \frac{1}{1 + \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{s}_j}, \\ b_{12}(z) &= \frac{1}{1 + n^{-1} E \text{tr} \mathbf{T} \mathbf{A}_{12}^{-1}(z)}, \quad \beta_{kj}^{tr}(z) = \frac{1}{1 + n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z)}\end{aligned}$$

and

$$\xi_{kj}(z) = \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{s}_j - E n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{T}, \quad \eta_{kj}(z) = \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{s}_j - n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{T}.$$

Note that

$$(2.24) \quad \mathbf{A}_k^{-1}(z) - \mathbf{A}_{kj}^{-1}(z) = -\beta_{kj}(z) \mathbf{A}_{kj}^{-1}(z) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z)$$

and (see Lemma 2.10 of [1]) for any $p \times p$ matrix \mathbf{D}

$$(2.25) \quad |\text{tr}(\mathbf{A}_k^{-1}(z) - \mathbf{A}_{kj}^{-1}(z)) \mathbf{D}| \leq \frac{\|\mathbf{D}\|}{v}.$$

By Lemma 3 in Appendix 1 and (2.25) we have

$$(2.26) \quad \frac{1}{n^4} E |\text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{T} - E \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{T}|^8 \leq \frac{M}{n^8 v^{12} \|\mathbf{T}\|^8}, \quad E |\xi_{kj}(z)|^8 \leq \frac{M}{n^4 v^4}.$$

By Lemma 3 we have

$$(2.27) \quad E \frac{1}{n} \text{tr} \mathbf{A}^{-1}(z) \mathbf{A}^{-1}(\bar{z}) = \frac{1}{v} \text{Im}(E \frac{1}{n} \text{tr} \mathbf{A}^{-1}(z)) \leq \frac{M}{v},$$

which, together with (2.25), implies that

$$(2.28) \quad E \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{A}_{kj}^{-1}(\bar{z}) \leq \frac{M}{v}.$$

By (2.25) and the fact that $b_1(z)$ is bounded, given in Lemma 3, it is straightforward to verify that $|b_1(z) - b_{12}(z)| \leq \frac{1}{nv^2}$ and hence

$$(2.29) \quad |b_{12}(z)| \leq M.$$

We are now in a position to state Lemma 1:

Lemma 1. *For non-random matrix \mathbf{D}*

$$(2.30) \quad \left| E \left[\frac{1}{n} \text{tr} \mathbf{D} \mathbf{A}_k^{-1}(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \right] - E \left(\frac{1}{n} \text{tr} \mathbf{D} \mathbf{A}_k^{-1}(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \right) \right|^2 \leq \begin{cases} \frac{M}{n^2 v^5} & \text{when } \|\mathbf{D}\| \leq M \\ \frac{M}{n^2 v^6} & \text{when } \frac{1}{n} \text{tr} \mathbf{D} \mathbf{D}^* \leq M. \end{cases}$$

Remark 4. *Checking on the argument of Lemma 1, we see that Lemma 1 holds as well when we replace $E_k(\mathbf{A}_k^{-1}(z_2))$ by $\mathbf{A}_k^{-1}(z_2)$. The main difference of arguments is that we do not need to distinguish between the cases $j < k$ and $j > k$ when dealing with the latter.*

Proof. We begin with a martingale decomposition of random variable of interest:

$$\begin{aligned} & \frac{1}{n} \text{tr} \mathbf{D} \mathbf{A}_k^{-1}(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) - E \left(\frac{1}{n} \text{tr} \mathbf{D} \mathbf{A}_k^{-1}(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \right) \\ &= \frac{1}{n} \sum_{j \neq k}^n (E_j - E_{j-1}) \left[\text{tr} \mathbf{D} \mathbf{A}_k^{-1}(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \right] \\ &= \frac{1}{n} \sum_{j \neq k}^n (E_j - E_{j-1}) \left[\text{tr} \mathbf{D} \mathbf{A}_k^{-1}(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) - \text{tr} \mathbf{D} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} E_k(\mathbf{A}_{kj}^{-1}(z_2)) \right] \\ &= \frac{1}{n} \sum_{j \neq k}^n (E_j - E_{j-1}) (\delta_1 + \delta_2 + \delta_3), \end{aligned}$$

where, via (2.24),

$$\delta_1 = \beta_{kj}(z_1) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} E_k \left(\beta_{kj}(z_2) \mathbf{A}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_2) \right) \mathbf{D} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{s}_j$$

$$\delta_2 = -\beta_{kj}(z_1) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} E_k \left(\mathbf{A}_{kj}^{-1}(z_2) \right) \mathbf{D} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{s}_j$$

and

$$\delta_3 = -\text{tr} \mathbf{D} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} E_k \left(\beta_{kj}(z_2) \mathbf{A}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_2) \right).$$

Note that

$$(2.31) \quad |\beta_{kj}| \|\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z)\|^2 = |\beta_{kj} \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{A}_{kj}^{-1}(\bar{z}) \mathbf{s}_j| \leq \frac{1}{v}$$

which implies that

$$(2.32) \quad |\delta_1| \leq \frac{1}{v^2 \|\mathbf{D}\|}.$$

Write

(2.33)

$$\beta_{kj}(z) = b_{12}(z) - \beta_{kj}(z)b_{12}(z)\xi_{kj}(z) = b_{12}(z) - b_{12}^2(z)\xi_{kj}(z) + \beta_{kj}(z)b_{12}^2(z)\xi_{kj}^2(z).$$

This implies that when $j > k$,

$$(E_j - E_{j-1})\delta_1 = (E_j - E_{j-1})b_{12}(z_1)(\delta_{11} - \delta_{12}),$$

where $\delta_{12} = \xi_{kj}(z_1)\delta_1$ and

$$\delta_{11} = \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} E_k \left(\beta_{kj}(z_2) \mathbf{G}_k(z_2) \right) \mathbf{D} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{s}_j$$

$$- n^{-1} \text{tr} \mathbf{T} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} E_k \left(\beta_{kj}(z_2) \mathbf{G}_k(z_2) \right) \mathbf{D} \mathbf{A}_{kj}^{-1}(z_1)$$

with $\mathbf{G}_k(z_2) = \mathbf{A}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_2)$. When $\|\mathbf{D}\| \leq M$ we conclude from (2.31), (2.32), (2.28) and Lemma 4 that

$$E \left| \frac{1}{n} \sum_{j \neq k}^n (E_j - E_{j-1})(\delta_{11} + \delta_{12}) \right|^2 \leq \frac{1}{n^2} \sum_{j \neq k}^n E |\delta_{11}|^2 + E |\delta_{12}|^2 \leq \frac{M}{n^2 v^5 \|\mathbf{D}\|^2}.$$

When $\frac{1}{n} \text{tr} \mathbf{D} \mathbf{D}^* \leq M$, by Lemma 4 and (2.31)

$$E |\delta_{11}|^8 \leq \frac{M}{n^4 v^{24}} \left(\frac{1}{n} \text{tr} \mathbf{D} \mathbf{D}^* \right)^4 \leq \frac{M}{n^4 v^{24}}.$$

This, together with (2.26) and Lemma 3, implies

$$\begin{aligned} E |\delta_{12}|^2 &\leq M E |\xi_{kj} \beta_{kj} \delta_{11}|^2 + \frac{M}{n^2} E |\xi_{kj} \beta_{kj} \text{tr} \mathbf{T} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} E_k \left(\beta_{kj}(z_2) \mathbf{G}_k(z_2) \right) \mathbf{D} \mathbf{A}_{kj}^{-1}(z_1)|^2 \\ &\leq \frac{M}{n v^6}, \end{aligned}$$

because via (2.31) and Holder's inequality

$$\begin{aligned} &\frac{1}{n^4} E |\text{tr} \mathbf{T} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} E_k \left(\beta_{kj}(z_2) \mathbf{G}_k(z_2) \right) \mathbf{D} \mathbf{A}_{kj}^{-1}(z_1)|^8 \\ &\leq \frac{M}{v^{16}} E \left(\frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{A}_{kj}^{-1}(\bar{z}_1) \right)^4 \left(\frac{1}{n} \text{tr} \mathbf{D} \mathbf{D}^* \right)^4 \\ (2.34) &\leq \frac{M}{v^{20}} E \left| \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) - E \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \right|^4 + \frac{M}{v^{20}} \left| E \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \right|^4 \leq \frac{M}{v^{20}}. \end{aligned}$$

These give

$$E \left| \frac{1}{n} \sum_{j \neq k}^n (E_j - E_{j-1})(\delta_{11} + \delta_{12}) \right|^2 \leq \frac{1}{n^2} \sum_{j \neq k}^n E |\delta_{11}|^2 + E |\delta_{12}|^2 \leq \frac{M}{n^2 v^6}.$$

For handling the case $j < k$, we define $\underline{\mathbf{A}}_{kj}^{-1}(z), \underline{\beta}_{kj}(z)$ and $\underline{\xi}_{kj}(z)$ using $\mathbf{s}_1, \dots, \mathbf{s}_{j-1}, \underline{\mathbf{s}}_{j+1}, \dots, \underline{\mathbf{s}}_{k-1}, \underline{\mathbf{s}}_{k+1}, \dots, \underline{\mathbf{s}}_n$ as $\mathbf{A}_{kj}^{-1}(z), \beta_{kj}(z)$ and $\xi_{kj}(z)$ are defined using $\mathbf{s}_1, \dots, \mathbf{s}_{j-1}, \mathbf{s}_{j+1}, \dots, \mathbf{s}_{k-1}, \mathbf{s}_{k+1}, \dots, \mathbf{s}_n$. Here $\underline{\mathbf{s}}_1, \dots, \underline{\mathbf{s}}_n$ are i.i.d. copies of \mathbf{s}_1 and independent of $\{\mathbf{s}_j, j = 1, \dots, n\}$. Let

$$\alpha_{k1} = \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j, \quad \alpha_{k2} = \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{D} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j.$$

Applying (2.33) and the equality for $\underline{\beta}_{kj}(z_2)$ similar to (2.33) yields

$$\begin{aligned} (E_j - E_{j-1})\delta_1 &= (E_j - E_{j-1}) \left[\beta_{kj}(z_1) \underline{\beta}_{kj}(z_2) \alpha_{k1} \alpha_{k2} \right] \\ &= b_{12}(z_1) b_{12}(z_2) [\delta_{13} + \delta_{14} + \delta_{15} + \delta_{16} + \delta_{17} + \delta_{18}], \end{aligned}$$

where

$$\begin{aligned} \delta_{13} &= (E_j - E_{j-1}) \left(\zeta_{kj1} \zeta_{kj2} \right), \delta_{14} = (E_j - E_{j-1}) \left(\zeta_{kj1} n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{D} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{T} \right), \\ \delta_{15} &= (E_j - E_{j-1}) \left(\zeta_{kj2} n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{T} \right), \\ \delta_{16} &= -(E_j - E_{j-1}) \left[\beta_{kj}(z_1) \xi_{kj}(z_1) \alpha_{k1} \alpha_{k2} \right], \\ \delta_{17} &= -(E_j - E_{j-1}) \left[\underline{\beta}_{kj}(z_2) \underline{\xi}_{kj}(z_2) \alpha_{k1} \alpha_{k2} \right] \end{aligned}$$

and

$$\delta_{18} = (E_j - E_{j-1}) \left[\beta_{kj}(z_1) \underline{\beta}_{kj}(z_2) \xi_{kj}(z_1) \underline{\xi}_{kj}(z_2) \alpha_{k1} \alpha_{k2} \right],$$

with

$$\zeta_{kj1} = \alpha_{k1} - n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{T}, \quad \zeta_{kj2} = \alpha_{k2} - n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{D} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{T}.$$

Consider $\|\mathbf{D}\| \leq M$ first. It follows from Lemma 4, (2.26) and (2.28) that (or see (5.8) in Appendix 1)

$$(2.35) \quad E|\zeta_{kj1}|^4 \leq \frac{M}{n^2 v^4} E \left| \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{A}_{kj}^{-1}(\bar{z}_1) \right|^2 \leq \frac{M}{n^2 v^6}, \quad E|\zeta_{kj2}|^4 \leq \frac{M}{n^2 v^6 \|\mathbf{D}\|^2}.$$

Similarly, by (2.26) and (2.28), as in (2.34), we have

$$\begin{aligned} & E |n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{D} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{T}|^4 \\ & \leq \frac{M}{\|\mathbf{D}\|^4} \left(E |n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{A}_{kj}^{-1}(\bar{z}_1)|^4 E |n^{-1} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \underline{\mathbf{A}}_{kj}^{-1}(\bar{z}_2)|^4 \right)^{1/2} \\ (2.36) & \leq \frac{M}{\|\mathbf{D}\|^4 v^4}. \end{aligned}$$

In view of (2.35) and (2.29),

$$E \left| \frac{1}{n} \sum_{j \neq k}^n (E_j - E_{j-1}) b_{12}(z_1) b_{12}(z_2) (\delta_{13}) \right|^2 \leq \frac{M}{n^3 v^6 \|\mathbf{D}\|^2}.$$

While (2.36) and (2.29) yield

$$E\left|\frac{1}{n}\sum_{j \neq k}^n (E_j - E_{j-1})b_{12}(z_1)b_{12}(z_2)(\delta_{1j})\right|^2 \leq \frac{M}{n^2 v^5 \|\mathbf{D}\|^2}, \quad j = 4, 5.$$

It follows from (2.31) that

$$|\beta_{kj}(z_1)\alpha_{k1}\alpha_{k2}| \leq \frac{M}{v} \|\underline{\mathbf{A}}_{kj}^{-1}(z_2)\mathbf{s}_j\|^2 = \frac{M}{v} \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(\bar{z}_2) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j.$$

In the mean time we obtain from Lemma 4

$$(2.37) \quad E|\mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(\bar{z}_2) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j - \frac{1}{n} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(\bar{z}_2) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{T}|^8 \leq \frac{M}{n^4 v^{12}}.$$

Thus we have

$$E\left|\frac{1}{n}\sum_{j \neq k}^n (E_j - E_{j-1})b_{12}(z_1)b_{12}(z_2)(\delta_{16})\right|^2 \leq \frac{M}{n^2 v^5}.$$

Obviously, this estimate applies to the term involving δ_{17} . From (2.31) and Lemma 3 we obtain

$$E\left|\frac{1}{n}\sum_{j \neq k}^n (E_j - E_{j-1})b_{12}(z_1)b_{12}(z_2)(\delta_{18})\right|^2 \leq \frac{M}{n^3 v^6 \|\mathbf{D}\|^2}.$$

Summarizing the above we have

$$E\left|\frac{1}{n}\sum_{j \neq k}^n (E_j - E_{j-1})b_{12}(z_1)b_{12}(z_2)(\delta_1)\right|^2 \leq \frac{M}{n^2 v^5 \|\mathbf{D}\|^2}.$$

Consider $\frac{1}{n} \text{tr} \mathbf{D} \mathbf{D}^* \leq M$ next. By Lemma 4

$$(2.38) \quad E|\zeta_{kj2}|^8 \leq \frac{M}{n^4} E\left(\frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{D} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(\bar{z}_2) \mathbf{D} \mathbf{A}_{kj}^{-1}(\bar{z}_1) \mathbf{T}\right)^4 \leq \frac{M}{n^4 v^{16}} \left(\frac{1}{n} \text{tr} \mathbf{D} \mathbf{D}\right)^2.$$

Observe that

$$(2.39) \quad E|n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{D} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{T}|^8 \leq \frac{M}{v^{12}} |n^{-1} \text{tr} \mathbf{D} \mathbf{D}^*|^4.$$

This, together with (2.35) and (2.36), give

$$E\left|\frac{1}{n}\sum_{j \neq k}^n (E_j - E_{j-1})b_{12}(z_1)b_{12}(z_2)(\delta_{13})\right|^2 \leq \frac{M}{n^3 v^7},$$

and

$$E\left|\frac{1}{n}\sum_{j \neq k}^n (E_j - E_{j-1})b_{12}(z_1)b_{12}(z_2)(\delta_{1j})\right|^2 \leq \frac{M}{n^2 v^6}, \quad j = 4, 5.$$

To deal with δ_{16} , we obtain from (2.31)

$$\begin{aligned} E|\beta_{kj}(z_1)\xi_{kj}(z_1)\alpha_{k1}\alpha_{k2}|^2 &\leq \frac{M}{v}E\left(\sqrt{|\beta_{kj}(z_1)|}\xi_{kj}(z_1)\zeta_{kj2}\|\underline{\mathbf{A}}_{kj}^{-1}(z_2)\mathbf{s}_j\|\right)^2 \\ &+ \frac{M}{v}E\left(\sqrt{|\beta_{kj}(z_1)|}\xi_{kj}(z_1)\|\underline{\mathbf{A}}_{kj}^{-1}(z_2)\mathbf{s}_j\|n^{-1}tr\mathbf{A}_{kj}^{-1}(z_1)\mathbf{D}\underline{\mathbf{A}}_{kj}^{-1}(z_2)\mathbf{T}\right)^2. \end{aligned}$$

We conclude from (2.38), (2.26), (2.37) and Lemma 3

$$\begin{aligned} &E\left(\sqrt{|\beta_{kj}(z_1)|}\xi_{kj}(z_1)\zeta_{kj2}\|\underline{\mathbf{A}}_{kj}^{-1}(z_2)\mathbf{s}_j\|\right)^2 \\ &\leq (E|\xi_{kj}|^8 E|\zeta_{kj2}|^8)^{1/4} (E|\beta_{kj}|^4 E|\mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(\bar{z}_2) \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j|^4)^{1/4} \leq \frac{M}{n^2 v^6}. \end{aligned}$$

Replacing ζ_{kj2} with $n^{-1}tr\mathbf{A}_{kj}^{-1}(z_1)\mathbf{D}\underline{\mathbf{A}}_{kj}^{-1}(z_2)\mathbf{T}$ in the above similarly gives

$$E\left(\sqrt{|\beta_{kj}(z_1)|}\xi_{kj}(z_1)\|\underline{\mathbf{A}}_{kj}^{-1}(z_2)\mathbf{s}_j\|n^{-1}tr\mathbf{A}_{kj}^{-1}(z_1)\mathbf{D}\underline{\mathbf{A}}_{kj}^{-1}(z_2)\mathbf{T}\right)^2 \leq \frac{M}{nv^5}.$$

Consequently $E|\frac{1}{n}\sum_{j \neq k}^n (E_j - E_{j-1})b_{12}(z_1)b_{12}(z_2)(\delta_{16})|^2 \leq \frac{M}{n^2 v^6}$. This estimate apparently applies to the term involving δ_{17} . Also, for δ_{18} we similarly have

$$E|\delta_{18}|^2 \leq \frac{M}{v^3}E\left(\sqrt{|\beta_{kj}(z_2)|}\xi_{kj}(z_1)\xi_{kj}(z_1)\|\underline{\mathbf{A}}_{kj}^{-1}(z_2)\mathbf{s}_j\|\right)^2 \leq \frac{M}{n^2 v^6}.$$

The terms δ_2 and δ_3 can be similarly, even simpler, proved to have the same order. Thus Lemma 1 is complete. \square

Combining (1.8), (2.18), (2.20), (2.23) and Lemma 1 with $\mathbf{D} = I$ we conclude that

$$\begin{aligned} &-\frac{1}{4\pi^2}\sum_{k=1}^n E_{k-1}[Y_k(x_1)Y_k(x_2)] \\ (2.40) = &-\frac{1}{2h^2\pi^2}\int\int K'(\frac{x_1-z_1}{h})K'(\frac{x_2-z_2}{h})a_{n1}(z_1, z_2)dz_1dz_2 + o_p(1), \end{aligned}$$

where

$$a_{n1}(z_1, z_2) = \frac{b_1(z_1)b_1(z_2)}{n^2}\sum_{k=1}^n E\left[tr\mathbf{T}\mathbf{A}_k^{-1}(z_1)\mathbf{T}E_k(\mathbf{A}_k^{-1}(z_2))\right].$$

Thus, it is enough to investigate the uniform convergence of $a_{n1}(z_1, z_2)$.

2.1. The limit of $a_{n1}(z_1, z_2)$. Before developing the limit of $a_{n1}(z_1, z_2)$, we first establish Lemma 2 below, which improves (5.1) in Lemma 3.

Lemma 2.

$$(2.41) \quad \frac{1}{n^2} E |tr \mathbf{A}_k^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} - E tr \mathbf{A}_k^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D}|^2 \leq \frac{M}{n^2 v^3 \|\mathbf{D}\|^2},$$

where $\mathbf{F}^{-1}(z) = (E \underline{\mathbf{m}}_n \mathbf{T} + \mathbf{I})^{-1}$ and $\mathbf{D} = \mathbf{T}$ or $(z\mathbf{I} - \frac{n-1}{n} b_{12}(z) \mathbf{T})^{-1}$.

Proof. With notation $\mathbf{F}_1^{-1}(z) = (z\mathbf{I} - \frac{n-1}{n} b_{12}(z) \mathbf{T})^{-1}$, we start the proof of Lemma 2 by presenting the equality (2.9) in [2]

$$(2.42) \quad \mathbf{A}_k^{-1}(z) = -\mathbf{F}_1^{-1}(z) + b_{12}(z) B(z) + C(z) + D(z),$$

where

$$B(z) = \sum_{j \neq k} \mathbf{F}_1^{-1}(z) (\mathbf{s}_j \mathbf{s}_j^T - n^{-1} \mathbf{T}) \mathbf{A}_{kj}^{-1}(z),$$

$$C(z) = \sum_{j \neq k} (\beta_{kj}(z) - b_{12}(z)) \mathbf{F}_1^{-1}(z) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z)$$

and

$$D(z) = n^{-1} b_{12}(z) \mathbf{F}_1^{-1}(z) \mathbf{T} \sum_{j \neq k} (\mathbf{A}_{kj}^{-1}(z) - \mathbf{A}_k^{-1}(z)).$$

By (1.11) we have

$$(2.43) \quad \frac{1}{n} tr \mathbf{F}_1^{-2}(z) \mathbf{F}_1^{-2}(\bar{z}) \leq M, \quad \frac{1}{n} tr \mathbf{F}^{-2}(z) \mathbf{F}^{-2}(\bar{z}) \leq M.$$

However sometimes we also use the fact that (see (2.10) in [2] and Lemma 2.11 of [1])

$$(2.44) \quad \|\mathbf{F}_1(z)\| \leq \frac{M}{v}, \quad \|\mathbf{F}(z)\| \leq \frac{M}{v}.$$

With $\mathbf{H} = \mathbf{F}^{-1}(z)$ or $\mathbf{H} = \mathbf{I}$, we first apply (2.42) with z replaced by z_2 to prove that

$$(2.45) \quad \frac{1}{n} E \left[tr \mathbf{A}_k^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right] = -\frac{1}{n} tr \mathbf{F}_1^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} + O\left(\frac{1}{nv^{5/2}}\right),$$

which, together with (2.43), then implies that

$$(2.46) \quad \left| \frac{1}{n} E \left[tr \mathbf{A}_k^{-1}(z) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T} \right] \right| \leq M.$$

To see (2.45), first note that

$$\frac{1}{n} E \left[tr B(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right] = 0.$$

Second, applying (2.33) yields

$$\begin{aligned} & \frac{1}{n} E \left[\text{tr} C(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right] = -\frac{1}{n} \sum_{j \neq k} E \left[b_{12}^2(z_1) \eta_{kj}(z_1) \eta_{kj1} \right] \\ & - \frac{1}{n} \sum_{j \neq k} E \left[b_{12}^2(z_1) \frac{1}{n} (\text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{T} - E \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{T}) \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T} \right] \\ & + \frac{1}{n} \sum_{j \neq k} E \left[\beta_{kj}(z) b_{12}^2(z) \xi_{kj}^2 \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{s}_j \right], \end{aligned}$$

where

$$\eta_{kj1} = \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{s}_j - \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T}.$$

By Lemma 4, (2.43) and (2.44)

$$(2.47) \quad E |\eta_{kj1}|^4 \leq \frac{M}{n^2 v^8} \left(\frac{1}{n} \text{tr} \mathbf{F}_1^{-2}(z) \mathbf{F}_1^{-2}(\bar{z}) \right)^2 \leq \frac{M}{n^2 v^8},$$

and via Holder's inequality and (2.43)

$$(2.48) \quad \left| \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T} \right| \leq \frac{M}{v}.$$

Appealing to (2.47) and (2.26) yields

$$\left| E \left[b_{12}(z_1) \xi_{kj}(z_1) \eta_{kj1} \right] \right| \leq \frac{M}{n v^{5/2}}.$$

We obtain from (2.48) and (2.26)

$$\left| E \left[\frac{1}{n} (\text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{T} - E \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{T}) \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T} \right] \right| \leq \frac{M}{n v^{5/2}}.$$

In view of (2.47), (2.26), (2.48) and Lemma 3 we have

$$\begin{aligned} & \left| E \left[\beta_{kj}(z_1) \xi_{kj}^2 \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{s}_j \right] \right| \\ & \leq (E |\xi_{kj}(z_1)|^4)^{1/2} (E |\beta_{kj}(z)|^4 E |\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{s}_j|^4)^{1/4} \leq \frac{M}{n v^{5/2}}. \end{aligned}$$

Let η_{kj2} equal to

$$\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{A}_{kj}^{-1}(z) \mathbf{s}_j - \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{A}_{kj}^{-1}(z) \mathbf{T}.$$

Then, as in (2.47),

$$(2.49) \quad E |\eta_{kj2}|^4 \leq \frac{M}{n^2 v^{12}}, \quad \left| \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{A}_{kj}^{-1}(z) \mathbf{T} \right| \leq \frac{M}{v^2}.$$

This, together with (2.33), (2.26) and Lemma 3, ensures that

$$\left| \frac{1}{n} E \left[\text{tr} D(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right] \right| \leq \frac{M}{n^{3/2} v^{5/2}}.$$

Thus (2.45) is true.

When replacing $\mathbf{F}_1^{-1}(z)$, \mathbf{T} with $\mathbf{F}^{-1}(\bar{z})$, \mathbf{I} , respectively, (2.46) further ensures that

$$(2.50) \quad E \frac{1}{n} \text{tr} \mathbf{F}^{-1}(\bar{z}) \mathbf{A}_k^{-1}(\bar{z}) \mathbf{A}_k^{-1}(z) \mathbf{F}^{-1}(z) = \frac{1}{v} \Im \left(E \frac{1}{n} \text{tr} \mathbf{F}^{-1}(\bar{z}) \mathbf{A}_k^{-1}(z) \mathbf{F}^{-1}(z) \right) \leq \frac{M}{v}.$$

As before, we get a martingale representation as follows:

$$\begin{aligned} & \frac{1}{n} \text{tr} \mathbf{A}_k^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} - E \text{tr} \mathbf{A}_k^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \\ &= \frac{1}{n} \sum_{j \neq k} (E_j - E_{j-1}) \left(\text{tr} \mathbf{A}_k^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} - \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \right) \\ &= \frac{1}{n} \sum_{j \neq k} (E_j - E_{j-1}) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \mathbf{A}_{kj}^{-1}(z) \mathbf{s}_j \beta_{kj} \\ &= \frac{b_{12}(z)}{n} \sum_{j \neq k} (E_j - E_{j-1}) \left(\phi_1 + \phi_2 + \phi_3 \right), \end{aligned}$$

where

$$\begin{aligned} \phi_1 &= \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \mathbf{A}_{kj}^{-1}(z) \mathbf{s}_j - \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \mathbf{A}_{kj}^{-1}(z) \mathbf{T}, \\ \phi_2 &= -b_{12}(z) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \mathbf{A}_{kj}^{-1}(z) \mathbf{s}_j \xi_{kj}(z), \end{aligned}$$

and

$$\phi_3 = b_{12}(z) \beta_{kj} \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \mathbf{A}_{kj}^{-1}(z) \mathbf{s}_j \xi_{kj}^2(z).$$

Here in the last step one uses (2.33). By Lemma 4 and (2.50) we have

$$\begin{aligned} E|\phi_1|^2 &\leq \frac{M}{n^2} E \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \mathbf{A}_{kj}^{-1}(z) \mathbf{T} \mathbf{A}_{kj}^{-1}(\bar{z}) \mathbf{D} \mathbf{F}^{-1}(\bar{z}) \mathbf{A}_{kj}^{-1}(\bar{z}) \\ (2.51) \quad &\leq \frac{M}{n^2 v^2 \|\mathbf{D}\|^2} E \text{tr} \mathbf{F}^{-1}(\bar{z}) \mathbf{A}_{kj}^{-1}(\bar{z}) \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \leq \frac{M}{n v^3 \|\mathbf{D}\|^2}. \end{aligned}$$

Similarly, from (2.43) and (2.28) it is easy to verify that

$$(2.52) \quad E|\phi_1|^8 \leq \frac{M}{n^4 v^{14} \|\mathbf{D}\|^8}.$$

As in (2.13), by Lemma 3, (2.26) and (2.50) we have

$$\begin{aligned} & E \left| \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \mathbf{A}_{kj}^{-1}(z) \mathbf{T} \eta_{kj}(z) \right|^2 \\ &\leq \frac{M}{n} E \left(\left| \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \mathbf{A}_{kj}^{-1}(z) \mathbf{T} \right|^2 \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{A}_{kj}^{-1}(\bar{z}) \right) \\ &\leq \frac{M}{n \|\mathbf{D}\|^2} E \left[\frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{F}^{-1}(\bar{z}) \mathbf{A}_{kj}^{-1}(\bar{z}) \left(\frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{A}_{kj}^{-1}(\bar{z}) \right)^2 \right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{M}{n^3 v^2 \|\mathbf{D}\|^2} E \left[\text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{F}^{-1}(\bar{z}) \mathbf{A}_{kj}^{-1}(\bar{z}) \right. \\
(2.53) \quad &\left. \times \left(\left| \text{tr} \mathbf{A}_{kj}^{-1}(z) - E \text{tr} \mathbf{A}_{kj}^{-1}(z) \right|^2 + |E \text{tr} \mathbf{A}_{kj}^{-1}(z)|^2 \right) \right] \leq \frac{M}{n v^3 \|\mathbf{D}\|^2}.
\end{aligned}$$

Note that

$$(E_j - E_{j-1}) \left[\frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \mathbf{A}_{kj}^{-1}(z) \mathbf{T} \left(\frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{T} - \frac{1}{n} E \mathbf{A}_{kj}^{-1}(z) \mathbf{T} \right) \right] = 0.$$

This, together with (2.52), (2.53) and (2.26), implies that

$$E \left| \frac{1}{n} \sum_{j \neq k} (E_j - E_{j-1}) \phi_2 \right|^2 \leq \frac{M}{n^2 v^3 \|\mathbf{D}\|^2}.$$

By (2.52) and Lemma 3 we obtain

$$(2.54) \quad (E |\xi_{kj}(z)|^8)^{1/2} (E |\phi_1|^8 E |\beta_{kj}|^8)^{1/4} \leq \frac{M}{n^3 v^{13/2}} (E |\beta_{kj}|^4)^{1/4} \leq \frac{M}{n^3 v^{13/2}}.$$

From Holder's inequality, (2.28) and (2.43)

$$E \left| \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \mathbf{A}_{kj}^{-1}(z) \mathbf{T} \right|^8 \leq E \left| \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-2}(z) \mathbf{A}_{kj}^{-2}(\bar{z}) \frac{1}{n} \text{tr} \mathbf{F}^{-1}(z) \mathbf{F}^{-1}(\bar{z}) \right|^4 \leq \frac{M}{v^{12}}.$$

This, together with Lemma 3 and (2.26), implies that

$$\begin{aligned}
&(E \left| \frac{1}{n} (\text{tr} \mathbf{A}_{kj}^{-1}(z) - E \text{tr} \mathbf{A}_{kj}^{-1}(z)) \right|^8)^{1/2} (E \left| \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \mathbf{A}_{kj}^{-1}(z) \mathbf{T} \right|^8 E |\beta_{kj}|^8)^{1/4} \\
&\leq \frac{M}{n^4 v^{10}}
\end{aligned}$$

and that

$$\begin{aligned}
&(E |\beta_{kj}|^4)^{1/2} (E \left| \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \mathbf{A}_{kj}^{-1}(z) \mathbf{T} \right|^4 |\eta_{kj}(z)|^8)^{1/2} \\
&\leq \frac{M}{n^2 v^2} (E \left| \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{D} \mathbf{A}_{kj}^{-1}(z) \mathbf{T} \frac{1}{n} \text{tr} \mathbf{A}_{kj}^{-1}(z) \right|^4)^{1/2} \leq \frac{M}{n^2 v^5}.
\end{aligned}$$

These and (2.54) ensure that

$$E |\phi_3|^2 \leq \frac{M}{n^2 v^5}.$$

Thus, Lemma 2 is complete. □

To later use, we now consider a more general form than $a_{n1}(z_1, z_2)$

$$(2.55) \quad a_{n2}^{(1)}(z_1, z_2) = \frac{1}{n^2} \sum_{k=1}^n E \left[\text{tr} \mathbf{T} \mathbf{A}_k^{-1}(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \mathbf{H} \right].$$

One should note that $a_{n2}^{(1)}(z_1, z_2)$ reduces to $a_{n2}(z_1, z_2)$ when $\mathbf{H} = \mathbf{I}$.

Applying the definition of $C(z_1)$ and (2.33) gives

$$(2.56) \quad \frac{1}{n} E \left[\text{tr} \mathbf{T} C(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \mathbf{H} \right] = C_1(z_1) + C_2(z_1),$$

where

$$C_1(z_1) = -b_{12}^2(z_1) \frac{1}{n} \sum_{j \neq k} E \left[\xi_{kj}(z_1) \mathbf{s}_j^T \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{s}_j \right]$$

and

$$C_2(z_1) = b_{12}^2(z_1) \frac{1}{n} \sum_{j \neq k} E \left[\beta_{kj}(z_1) \xi_{kj}^2(z) \mathbf{s}_j^T \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{s}_j \right].$$

Here

$$(2.57) \quad \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) = \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1).$$

Define

$$\zeta_{kj3} = \mathbf{s}_j^T \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{s}_j - \frac{1}{n} \text{tr} \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{T}.$$

Consider $j > k$ first. Then by Lemma 4 and (2.43)

$$(2.58) \quad E |\zeta_{kj3}|^4 \leq \frac{M}{n^2 v^8},$$

and via an argument similar to (2.36), (2.28) and Holder's inequality

$$(2.59) \quad E \left| \frac{1}{n} \text{tr} \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{T} \right|^4 \leq \frac{M}{v^6}.$$

It follows from (2.58), (2.59) and (2.26) that

$$\begin{aligned} |C_2(z_1)| &\leq \frac{M}{n} \sum_{j \neq k} (E |\xi_{kj}(z_1)|^4)^{1/2} \left[E |\beta_{kj}(z_1)|^4 \right. \\ &\quad \left. \times \left(E |\zeta_{kj3}|^4 + E \left| \frac{1}{n} \text{tr} \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{T} \right|^4 \right) \right]^{1/4} \leq \frac{M}{n v^{5/2}}. \end{aligned}$$

As for $C_1(z_1)$, by (2.58), (2.26), (2.25) and (2.30) with $\mathbf{D} = \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T}$ we have

$$\begin{aligned} &\frac{1}{n} \sum_{j > k} E \left[\xi_{kj}(z_1) \mathbf{s}_j^T \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{s}_j \right] = \frac{1}{n} \sum_{j > k} E \left[\eta_{kj}(z_1) \zeta_{kj3} \right. \\ &\quad \left. + \frac{1}{n^2} (\text{tr} \mathbf{A}_{kj}^{-1} - E \text{tr} \mathbf{A}_{kj}^{-1}) (\text{tr} \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{T} - E \text{tr} \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{T}) \right] = O\left(\frac{1}{n v^{5/2}}\right). \end{aligned}$$

When $j < k$, decompose $\mathbf{A}_k^{-1}(z_2)$ as

$$\mathbf{A}_{kj}^{-1}(z_2) - \mathbf{A}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_k^{-1}(z_2) \beta_{kj}.$$

Then, apparently, the above argument for the case $j > k$ also works if we replace $E_k(\mathbf{A}_k^{-1}(z_2))$ in $\hat{\mathbf{A}}_{kj}^{-1}(z_1, z_2)$ with $E_k(\mathbf{A}_{kj}^{-1}(z_2))$. For another term of $C_2(z_1)$ by (2.31), (2.26) and Lemma 3

$$\begin{aligned} & E \left| \beta_{kj}(z_1) \underline{\beta}_{kj}(z_2) \xi_{kj}^2(z) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{s}_j \right| \\ & \leq \frac{M}{v} (E |\beta_{kj}(z_1)|^2 E |\underline{\beta}_{kj}(z_2)|^2 E |\xi_{kj}|^8 E |\mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{s}_j|^4)^{1/4} \leq \frac{M}{nv^{5/2}}, \end{aligned}$$

because

$$(2.60) \quad E |\zeta_{kj4}|^4 \leq \frac{M}{n^2 v^4}, \quad E \left| \frac{1}{n} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right|^4 \leq \frac{M}{v^2},$$

with $\zeta_{kj4} = \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{s}_j - n^{-1} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T}$. As for another term of $C_1(z_1)$, an application of (2.33) yields

$$\begin{aligned} & \frac{1}{n} \sum_{j < k} E \left[\xi_{kj}(z_1) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} E_k \left(\mathbf{A}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_2) \beta_{kj}(z_2) \right) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{s}_j \right] \\ & = \frac{1}{n} \sum_{j < k} E \left[\underline{\beta}_{kj}(z_2) \xi_{kj}(z_1) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{s}_j \right] \\ (2.61) \quad & = \frac{b_{12}}{n} \sum_{j < k} [C_{11} + C_{12} + C_{13} + C_{14} + C_{15} + C_{16}], \end{aligned}$$

where

$$\begin{aligned} C_{11} &= E \left[\xi_{kj}(z_1) \zeta_{kj1} \zeta_{kj4} \right], \quad C_{12} = E \left[\xi_{kj}(z_1) \zeta_{kj1} n^{-1} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right] \\ C_{13} &= E \left[\xi_{kj}(z_1) \zeta_{kj4} n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{T} \right], \\ C_{14} &= E \left[\xi_{kj}(z_1) n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{T} n^{-1} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right] \\ &= \frac{1}{n} E \left[(\text{tr} \mathbf{A}_{kj}^{-1}(z_1)) \right. \\ &\quad \left. - E \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \right) n^{-1} \text{tr} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{T} n^{-1} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right] \\ C_{15} &= -E \left[\underline{\beta}_{kj}(z_2) \underline{\xi}_{kj}(z_2) \xi_{kj}(z_1) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j \zeta_{kj4} \right] \end{aligned}$$

and

$$C_{16} = -E \left[\underline{\beta}_{kj}(z_2) \underline{\xi}_{kj}(z_2) \xi_{kj}(z_1) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j n^{-1} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right].$$

Appealing to (2.36), (2.35), (2.26) and (2.60) yields

$$|C_{1j}| \leq \frac{M}{nv^{5/2}}, \quad j = 1, 2, 3.$$

By (2.46), (2.36), (2.26) and (2.41) we obtain $|C_{14}| \leq \frac{M}{nv^{5/2}}$. We conclude from (2.31), (2.26), (2.37), Lemma 3 and (2.60) that

$$\begin{aligned} |C_{15}| &\leq \frac{M}{\sqrt{v}} E \left| \sqrt{|\underline{\beta}_{kj}(z_2)|} \underline{\xi}_{kj}(z_2) \xi_{kj}(z_1) \|\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1)\| \zeta_{kj4} \right| \\ &\leq \frac{M}{\sqrt{v}} \left(E \left(\sqrt{|\underline{\beta}_{kj}(z_2)|} \|\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1)\| \right)^4 E |\underline{\xi}_{kj}(z_2)|^4 E |\xi_{kj}(z_1)|^4 E |\zeta_{kj4}|^4 \right)^{1/4} \\ &\leq \frac{M}{n^{3/2} v^{5/2}}. \end{aligned}$$

Similarly

$$|C_{16}| \leq \frac{M}{nv^{5/2}}.$$

Summarizing the above we have proved that

$$(2.62) \quad \left| \frac{1}{n} E \left[\text{tr} \mathbf{T} C(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \mathbf{H} \right] \right| \leq \frac{M}{nv^{5/2}}.$$

Consider $D(z_1)$ now. When $j > k$ using (2.33) and recalling the definition of $\hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2)$ in (2.57) we obtain

$$\frac{1}{n} E \left[\text{tr} \mathbf{T} D(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \mathbf{H} \right] = \frac{1}{n^2} b_{12}(z_1) \sum_{j \neq k} [D_1 + D_2 + D_3]$$

where

$$D_1 = -\frac{1}{n} E \left[\text{tr} \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{T} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \right], \quad D_2 = E \left[\zeta_{kj5} \xi_{kj}(z_1) \beta_{kj}(z_1) \right]$$

and

$$D_3 = \frac{1}{n} E \left[\text{tr} \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{T} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \xi_{kj}(z_1) \beta_{kj}(z_1) \right].$$

with

$$\zeta_{kj5} = \mathbf{s}_j^T \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{T} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{s}_j - \frac{1}{n} \text{tr} \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{T} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T},$$

Using (2.43), (2.25) and Holder's inequality

$$(2.63) \quad \frac{1}{n} E \left| \text{tr} \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{T} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \right|^2 \leq \frac{M}{v^5}.$$

By Lemma 2.7 [1] and (2.43)

$$(2.64) \quad E |\zeta_{kj5}|^2 \leq \frac{M}{nv^6}.$$

Thus

$$|D_1| \leq \frac{M}{v^{5/2}}, \quad |D_2| \leq \frac{M}{nv^{7/2}}, \quad |D_3| \leq \frac{M}{\sqrt{nv^3}}.$$

Hence when $j > k$

$$\left| \frac{1}{n} E \left[\text{tr} \mathbf{T} D(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \mathbf{H} \right] \right| \leq \frac{M}{nv^{5/2}}.$$

When $j < k$, divide $\mathbf{A}_k^{-1}(z_2)$ into the sum:

$$\mathbf{A}_{kj}^{-1}(z_2) - \mathbf{A}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_2) \beta_{kj}(z_2).$$

Apparently, the above argument for the case $j > k$ also works for the term involving $E_k(\mathbf{A}_{kj}^{-1}(z_2))$ if we replace $E_k(\mathbf{A}_k^{-1}(z_2))$ with $E_k(\mathbf{A}_{kj}^{-1}(z_2))$. Another term is

$$\begin{aligned} & \frac{b_{12}(z_1)}{n^2} \sum_{j \neq k} E \left[\beta_{kj}(z_1) \underline{\beta}_{kj}(z_2) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j \right. \\ & \quad \left. \times \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{s}_j \right], \end{aligned}$$

which has, via ((2.31), (2.38) and (2.39) with $\mathbf{D} = \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T}$, an order of $\frac{1}{nv^{5/2}}$.

Thus, the contribution from $C(z_1)$ and $D(z_1)$ is negligible.

Next consider $B(z_1)$. It follows from (2.24) that

$$\begin{aligned} (2.65) \quad & \frac{1}{n} E \left[\text{tr} \mathbf{T} B(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \mathbf{H} \right] \\ & = \frac{1}{n} \sum_{j < k} E \left[\mathbf{s}_j^T \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \mathbf{s}_j - n^{-1} \text{tr} \mathbf{T} \hat{\mathbf{A}}_{kjk}^{-1}(z_1, z_2) \right] = B_1(z_1) + B_2(z_1), \end{aligned}$$

where

$$B_1(z_1) = -\frac{1}{n} \sum_{j < k} E \left[\underline{\beta}_{kj}(z_2) \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{s}_j \right]$$

and

$$B_2(z_1) = -\frac{1}{n^2} \sum_{j < k} E \left[\underline{\beta}_{kj}(z_2) \mathbf{s}_j^T \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{s}_j \right].$$

For $B_2(z_1)$ we further write

$$B_2(z_1) = B_{21}(z_1) + B_{22}(z_1),$$

where

$$B_{21}(z_1) = -\frac{b_{12}(z_2)}{n^2} \sum_{j < k} E \left[\frac{1}{n} \text{tr} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \underline{\mathbf{A}}_{kj}^{-1}(z_2) \mathbf{T} \right],$$

and

$$B_{22}(z_1) = \frac{b_{12}(z_2)}{n^2} \sum_{j < k} E \left[\beta_{kj}(z_2) \xi_{kj}(z_2) \right. \\ \left. \times \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \mathbf{A}_{kj}^{-1}(z_2) \mathbf{s}_j \right].$$

The inequality similar to (2.63) ensures that $|B_{21}(z_1)| \leq \frac{M}{nv^{5/2}}$, while $|B_{22}(z_1)| \leq \frac{M}{n^{3/2}v^3}$ by estimates similar to (2.63) and (2.64). Therefore $B_2(z_1)$ is negligible.

By (2.33), (2.35) and the estimates of C_{1j} , $j = 1, 2, 3, 4$ in (2.61) we have

$$\left| B_1(z_1) + \frac{b_{12}(z_2)}{n} \sum_{j < k} E \left[\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \mathbf{A}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{s}_j \right] \right| \\ = O\left(\frac{M}{nv^{5/2}}\right).$$

In the mean time, (2.35) and (2.60) ensure that

$$\frac{1}{n} \sum_{j < k} E \left[\mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \mathbf{A}_{kj}^{-1}(z_2) \mathbf{s}_j \mathbf{s}_j^T \mathbf{A}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{s}_j \right] \\ = \frac{1}{n^3} \sum_{j < k} E \left[\text{tr} \mathbf{T} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \mathbf{A}_{kj}^{-1}(z_2) \text{tr} \mathbf{A}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right] + O\left(\frac{M}{nv^{5/2}}\right).$$

Furthermore we apply (2.30), (2.41), (2.25), (2.33) and (2.51) to obtain

$$\frac{1}{n^2} E \left[\text{tr} \mathbf{T} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \mathbf{A}_{kj}^{-1}(z_2) \text{tr} \mathbf{A}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right] \\ = \frac{1}{n^2} E \left[\text{tr} \mathbf{T} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \mathbf{A}_{kj}^{-1}(z_2) \right] E \left[\text{tr} \mathbf{A}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right] + O\left(\frac{M}{n^2 v^5}\right).$$

In addition, by (2.25), (2.38), (2.39), (2.33) and (2.46) we have

$$\frac{1}{n^2} E \left[\text{tr} \mathbf{T} \mathbf{A}_{kj}^{-1}(z_1) \mathbf{T} \mathbf{A}_{kj}^{-1}(z_2) \right] E \left[\text{tr} \mathbf{A}_{kj}^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right] \\ = \frac{1}{n^2} E \left[\text{tr} \mathbf{T} \mathbf{A}_k^{-1}(z_1) \mathbf{T} \mathbf{A}_k^{-1}(z_2) \right] E \left[\text{tr} \mathbf{A}_k^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right] + O\left(\frac{1}{nv^2}\right).$$

It follows that

$$(2.66) \quad \left| B_1(z_1) + \frac{j-1}{n^3} b_{12}(z_2) E \left(\text{tr} \mathbf{T} \mathbf{A}_k^{-1}(z_1) \mathbf{T} \mathbf{A}_k^{-1}(z_2) \right) \right. \\ \left. \times E \left(\text{tr} \mathbf{A}_k^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right) \right| \leq \frac{M}{nv^{5/2}}.$$

Summarizing the argument from (2.56) to (2.66) yields

$$(2.67) \quad \frac{1}{n} E \left[\text{tr} \mathbf{T} \mathbf{A}_k^{-1}(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \mathbf{H} \right] = -\frac{1}{n} E \left[\text{tr} \mathbf{A}_k^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right]$$

$$\begin{aligned}
& - \left[\frac{j-1}{n^3} b_{12}(z_1) b_{12}(z_2) E \left(\text{tr} \mathbf{T} \mathbf{A}_k^{-1}(z_1) \mathbf{T} \mathbf{A}_k^{-1}(z_2) \right) E \left(\text{tr} \mathbf{A}_k^{-1}(z_2) \mathbf{H} \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right) \right] \\
& + O\left(\frac{1}{nv^{5/2}}\right).
\end{aligned}$$

When $\mathbf{H} = \mathbf{I}$, (2.67) and (2.45) produce

$$\begin{aligned}
& \frac{1}{n} E \left[\text{tr} \mathbf{T} \mathbf{A}_k^{-1}(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \right] \left[1 - \frac{j-1}{n} b_{12}(z_1) b_{12}(z_2) \frac{1}{n} \text{tr} \mathbf{F}_1^{-1}(z_2) \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} \right] \\
(2.68) \quad & = \frac{1}{n} \text{tr} \mathbf{F}_1^{-1}(z_2) \mathbf{T} \mathbf{F}_1^{-1}(z_1) \mathbf{T} + O\left(\frac{1}{nv^{5/2}}\right).
\end{aligned}$$

By the formula (see (2.2) of [12])

$$\underline{m}_n(z) = -\frac{1}{zn} \sum_{k=1}^n \beta_k(z)$$

we have

$$(2.69) \quad E\beta_1(z) = -z E \underline{m}_n(z)$$

It follows from (2.6) that

$$|E\beta_1(z) - b_1(z)| = |b_1(z)^2 E(\beta_1(z) \xi_1^2(z))| \leq \frac{M}{nv}$$

and from (2.25) that

$$|b_1(z) - b_{12}(z)| \leq \frac{M}{nv}.$$

These, together with (3.8), imply that

$$(2.70) \quad |b_{12}(z) - \underline{m}_n^0(z)| \leq \frac{M}{nv^{3/2}}.$$

This, along with (2.68), (2.28) and (2.25), yields that

$$\begin{aligned}
& \frac{1}{n} E \left[\text{tr} \mathbf{T} \mathbf{A}_k^{-1}(z_1) \mathbf{T} E_k(\mathbf{A}_k^{-1}(z_2)) \right] \times \left[1 - \frac{j-1}{n} b_n(z_1, z_2) \right] \\
(2.71) \quad & = \frac{c_n b_n(z_1, z_2)}{z_1 z_2 \underline{m}_n^0(z_1) \underline{m}_n^0(z_2)} + O\left(\frac{1}{nv^{5/2}}\right),
\end{aligned}$$

where

$$b_n(z_1, z_2) = c_n \underline{m}_n^0(z_1) \underline{m}_n^0(z_2) \int \frac{t^2 dH_n(t)}{(1 + t \underline{m}_n^0(z_1))(1 + t \underline{m}_n^0(z_2))}.$$

It follows that

$$(2.72) \quad a_{n1}(z_1, z_2) = b_n(z_1, z_2) \frac{1}{n} \sum_{j=1}^n \frac{1}{1 - \frac{j-1}{n} b_n(z_1, z_2)} + O\left(\frac{1}{nv^{5/2}}\right).$$

From (2.19) in [2] and the inequality above (6.37) in [6] we see that
(2.73)

$$|1 - \frac{j-1}{n}b_n(z_1, z_2)| \geq Mv, \quad |1 - tb_n(z_1, z_2)| \geq Mv, \quad \text{for any } t \in [0, 1].$$

It follows that

$$|\frac{1}{n} \sum_{j=1}^n \frac{1}{1 - \frac{j-1}{n}b_n(z_1, z_2)} - \int_0^1 \frac{1}{1 - tb_n(z_1, z_2)}| \leq \frac{M}{nv^2}.$$

This ensures that

$$\begin{aligned} a_{n1}(z_1, z_2) &= b_n(z_1, z_2) \int_0^1 \frac{1}{1 - tb_n(z_1, z_2)} + O(\frac{1}{nv^{5/2}}) \\ &= -\log(1 - b_n(z_1, z_2)) + O(\frac{1}{nv^{5/2}}) \\ (2.74) \quad &= -\log\left((z_1 - z_2)\underline{m}_n^0(z_1)\underline{m}_n^0(z_2)\right) - \log(\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)) + O(\frac{1}{nv^{5/2}}), \end{aligned}$$

where in the last step one uses the fact that by (1.3)

$$z_1 - z_2 = \frac{\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)}{\underline{m}_n^0(z_1)\underline{m}_n^0(z_2)}(1 - b_n(z_1, z_2)).$$

So far we have considered $z \in \gamma_2$. The above argument evidently works for the case of $z \in \gamma_1$ due to symmetry. To deal with the cases when z belongs to two vertical lines of the contour, we need the estimates (1.9a) and (1.9b) of [2], which hold under our truncation level. That is

$$(2.75) \quad P(\|\mathbf{A}\| \geq \mu_1) = o(n^{-l}), \quad P(\lambda_{\min}^{\mathbf{A}} \leq \mu_2) = o(n^{-l}),$$

for any $\mu_1 > \limsup \mathbf{T}(1 + \sqrt{c})^2$, $\mu_2 < \liminf \mathbf{T}(1 - \sqrt{c})^2$ and l . This implies that

$$(2.76) \quad P(\|\mathbf{A}_k\| \geq \mu_1) = o(n^{-l}), \quad P(\lambda_{\min}^{\mathbf{A}_k} \leq \mu_2) = o(n^{-l}).$$

Let $B = \bigcap_{k=1}^n B_k$ where $B_k = (a_l - \eta < \lambda_{\min}^{\mathbf{A}_k} < \|\mathbf{A}_k\| < a_r - \eta)$ with $\eta > 0$ so that $a_r - \eta > \limsup \mathbf{T}(1 + \sqrt{c})^2$ and $a_l - \eta < \liminf \mathbf{T}(1 - \sqrt{c})^2$. Also define $B_{n+1} = (a_l - \eta < \lambda_{\min}^{\mathbf{A}} < \|\mathbf{A}\| < a_r - \eta)$ and let $C_k = B_k \cap B_{n+1}$. It follows that on the two vertical lines $\gamma_2 \cup \gamma_4$ (2.10) is equal to

$$\begin{aligned} &= -\frac{1}{h} \frac{1}{2\pi i} \sum_{k=1}^n (E_k - E_{k-1}) \int K'(\frac{x-z}{h}) \log \beta_k(z) dz \\ (2.77) \quad &= -\frac{1}{h} \frac{1}{2\pi i} \sum_{k=1}^n (E_k - E_{k-1}) \int K'(\frac{x-z}{h}) \log \beta_k(z) I(C_k) dz + o_p(1). \end{aligned}$$

We then introduce $\hat{\beta}_k(z)$, a truncated version of $\beta_k(z)$. Select a sequence of positive numbers ε_n satisfying for some $\beta \in (0, 1)$,

$$(2.78) \quad \varepsilon_n \downarrow 0, \quad \varepsilon_n \geq n^{-\beta}.$$

Define

$$\gamma_l = \{a_l + iv : v \in [n^{-1}\varepsilon_n, v_0h] \cup [-v_0h, -n^{-1}\varepsilon_n]\}$$

and

$$\gamma_r = \{a_r + iv : v \in [n^{-1}\varepsilon_n, v_0h] \cup [-v_0h, -n^{-1}\varepsilon_n]\}.$$

Write $\gamma_n = \gamma_r \cup \gamma_l$. We can now define the process

$$(2.79) \quad \hat{\beta}_k(z) = \begin{cases} \beta_k(z), & \text{if } z \in \gamma_n \\ \frac{nv+\varepsilon_n}{2\varepsilon_n}\beta_k(z_{r1}) + \frac{\varepsilon_n-nv}{2\varepsilon_n}\beta_k(z_{r2}), & \text{if } u = a_r, v \in [-n^{-1}\varepsilon_n, n^{-1}\varepsilon_n], \\ \frac{nv+\varepsilon_n}{2\varepsilon_n}\beta_k(z_{l1}) + \frac{\varepsilon_n-nv}{2\varepsilon_n}\beta_k(z_{l2}), & \text{if } u = a_l, v \in [-n^{-1}\varepsilon_n, n^{-1}\varepsilon_n], \end{cases}$$

where $z_{r1} = a_r + in^{-1}\varepsilon_n$, $z_{r2} = a_r - in^{-1}\varepsilon_n$, $z_{l1} = a_l + in^{-1}\varepsilon_n$, $z_{l2} = a_l - in^{-1}\varepsilon_n$. Note that $\|(\mathbf{A}_k - z\mathbf{I})^{-1}I(C_k)\| \leq \frac{1}{\eta}$, $\|(\mathbf{A} - z\mathbf{I})^{-1}I(C_k)\| \leq \frac{1}{\eta}$ and then $|\beta_k(z)I(C_k)| = |1 - \mathbf{s}_k^T(\mathbf{A} - z\mathbf{I})^{-1}\mathbf{s}_k I(C_k)| \leq M\mathbf{s}_k^T\mathbf{s}_k$. It follows that

$$(2.80) \quad P(|Q_k| \geq 1/2) \leq \frac{2\varepsilon_n E(\mathbf{s}_k^T\mathbf{s}_k)}{n} \rightarrow 0,$$

where $Q_k = \beta_k(z)(1/\beta_k(z) - 1/\hat{\beta}_k(z))I(C_k)$. By (2.15) and (2.4) we obtain

$$\begin{aligned} & \left| \frac{1}{h} \frac{1}{2\pi i} \sum_{k=1}^n (E_k - E_{k-1}) \oint K'(\frac{x-z}{h}) (\log \beta_k(z) - \log \hat{\beta}_k(z)) I(C_k) I(|Q_k| < \frac{1}{2}) dz \right| \\ & \leq \frac{M\varepsilon_n^2}{n^2} \sum_{k=1}^n (\mathbf{s}_k^T\mathbf{s}_k)^4 \xrightarrow{i.p.} 0. \end{aligned}$$

This, together with (2.77) and (2.80), ensures that

$$\begin{aligned} (2.10) &= \frac{1}{h} \frac{1}{2\pi i} \sum_{k=1}^n (E_k - E_{k-1}) \int K'(\frac{x-z}{h}) \log \hat{\beta}_k(z) I(C_k) dz + o_p(1) \\ &= \frac{1}{h} \frac{1}{2\pi i} \sum_{k=1}^n (E_k - E_{k-1}) \int K'(\frac{x-z}{h}) \log \left(\frac{\hat{\beta}_k^{tr}(z)}{\hat{\beta}_k(z)} \right) dz + o_p(1), \end{aligned}$$

where $\hat{\beta}_k^{tr}(z)$ is similarly defined according to $\hat{\beta}_k(z)$. Moreover, for the truncation versions, the higher moments of $\mathbf{A}^{-1}(z)$, $\mathbf{A}_k^{-1}(z)$ and $\mathbf{A}_{kj}^{-1}(z)$ are bounded (see (3.1) in [2]). Also, as pointed out in the paragraph below (3.2) in [2], the moments of $\beta_1(z)$, $\beta_{12}(z)$, $\beta^{tr}(z)$, $\mathbf{s}_1^T \mathbf{A}_1^{-1}(z_1) \mathbf{T} \mathbf{A}_1^{-1}(z_2) \mathbf{s}_1$ are bounded as well. Using these facts, all the estimates holding for $z \in \gamma_1 \cup \gamma_2$ also holds for the case where $z = z_{r1}, z_{r2}$ or $z \in \gamma_r \cup \gamma_l$. Via these facts, the arguments of the cases $z = z_{r1}, z_{r2}$ or $z \in \gamma_r \cup \gamma_l$, two vertical lines, can follow from those of

the case $z \in \gamma_1 \cup \gamma_2$ (here we omit the details) and hence their limits have the same form as (2.74).

In the mean time, appealing to Cauchy's theorem gives

$$(2.81) \quad \frac{1}{h^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} K'(\frac{x_1 - z_1}{h}) K'(\frac{x_2 - z_2}{h}) \log((z_1 - z_2) \underline{m}_n^0(z_1) \underline{m}_n^0(z_2)) dz_1 dz_2 = 0,$$

where the contour \mathcal{C}_2 is also a rectangle formed with four vertices $a_l - \varepsilon \pm 2iv_0h$ and $a_r + \varepsilon \pm 2iv_0h$ with $\varepsilon > 0$. One should note that the contour \mathcal{C}_2 encloses the contour \mathcal{C}_1 . Thus, in view of (2.74), it remains to find the limit of the following

$$(2.82) \quad - \frac{1}{2h^2\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} K'(\frac{x_1 - z_1}{h}) K'(\frac{x_2 - z_2}{h}) \log(\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)) dz_1 dz_2,$$

which is done in Appendix 3.

3. THE LIMIT OF MEAN FUNCTION

The aim in the section is to find the limit of

$$\frac{1}{2\pi i} \oint K(\frac{x - z}{h}) n(E\underline{m}_n(z) - \underline{m}_n^0(z)) dz.$$

It is thus sufficient to investigate the uniform convergence $nh(E\underline{m}_n(z) - \underline{m}_n^0(z))$ on the contour.

Recall that $\mathbf{F}^{-1}(z) = (E\underline{m}_n \mathbf{T} + \mathbf{I})^{-1}$ and then write (see (5.2) in [1])

$$(3.1) \quad n(c_n \int \frac{dH_n(t)}{1 + tE\underline{m}_n} + zc_n E(m_n(z))) = nD_n,$$

where

$$D_n = E\beta_1 \left[\mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{s}_1 - \frac{1}{n} E \left(\text{tr} \mathbf{F}^{-1}(z) \mathbf{T} \mathbf{A}^{-1}(z) \right) \right].$$

It follows that (see (3.20) in [1])

$$(3.2) \quad n(E\underline{m}_n(z) - \underline{m}_n^0(z)) = - \frac{n\underline{m}_n^0(z) D_n}{1 - c_n E\underline{m}_n \underline{m}_n^0 \int \frac{t^2 dH_n(t)}{(1 + tE\underline{m}_n)(1 + t\underline{m}_n^0)}}.$$

Considered $z \in \gamma_1 \cup \gamma_2$ first. Applying (2.6) and (2.5) yields

$$\begin{aligned}
& E\left(\text{tr}\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\right) - E\left(\text{tr}\mathbf{F}^{-1}(z)\mathbf{TA}^{-1}(z)\right) \\
&= E\left(\beta_1\mathbf{s}_1^T\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\mathbf{s}_1\right) \\
&= b_1E\left([1 - b_1\xi_1 + b_1\beta_1\xi_1^2(z)]\mathbf{s}_1^T\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\mathbf{s}_1\right) \\
&= b_1E\frac{1}{n}\text{tr}\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\mathbf{T} - d_{n1} + d_{n2} + d_{n3} \\
(3.3) \quad &= b_1E\frac{1}{n}\text{tr}\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\mathbf{T} + O\left(\frac{1}{nv^{5/2}}\right),
\end{aligned}$$

where

$$\begin{aligned}
d_{n1} &= b_1^2E\left[\eta_1(z)(\mathbf{s}_1^T\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\mathbf{s}_1 - \frac{1}{n}\text{tr}\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z))\right] \\
d_{n2} &= \frac{b_1^2}{n}E\left[\left(\text{tr}\mathbf{A}^{-1}(z)\mathbf{T} - E\text{tr}\mathbf{A}^{-1}(z)\mathbf{T}\right)\mathbf{s}_1^T\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\mathbf{s}_1\right] \\
&= \frac{b_1^2}{n^2}E\left[\left(\text{tr}\mathbf{A}^{-1}(z)\mathbf{T} - E\text{tr}\mathbf{A}^{-1}(z)\mathbf{T}\right)\right. \\
&\quad \left.\times\left(\text{tr}\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\mathbf{T} - E\text{tr}\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\mathbf{T}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
d_{n3} &= b_1E\left[\beta_1\xi_1^2(z)\mathbf{s}_1^T\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\mathbf{s}_1\right] \\
&= b_1E\left[\beta_1\xi_1^2(z)(\mathbf{s}_1^T\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\mathbf{s}_1 - \frac{1}{n}\text{tr}\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\mathbf{T})\right] \\
&\quad + b_1E\left[\beta_1\xi_1^2(z)\frac{1}{n}\text{tr}\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\mathbf{T}\right].
\end{aligned}$$

It follows from (2.51) and Lemma 3 that

$$|d_{nj}| \leq \frac{M}{nv^{5/2}}, \quad j = 1, 3,$$

where we also use the fact that

$$\begin{aligned}
& \left|\frac{1}{n}\text{tr}\mathbf{A}_1^{-1}(z)\mathbf{F}^{-1}(z)\mathbf{TA}_1^{-1}(z)\mathbf{T}\right| \\
(3.4) \quad & \leq \frac{M}{n}\left[\text{tr}\mathbf{F}^{-1}(z)\mathbf{F}^{-1}(\bar{z})\text{tr}(\mathbf{A}_1^{-1}(z)\mathbf{A}_1^{-1}(\bar{z}))^2\right]^{1/2} \leq \frac{M}{v^{3/2}}.
\end{aligned}$$

While, Lemma 3 and an estimate similar to (2.30) yield $|d_{n2}| \leq \frac{M}{nv^{5/2}}$.

Next by (2.6)

$$\begin{aligned}
& nE\left[\beta_1 \mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{s}_1\right] - E(\beta_1)E\left(\text{tr} \mathbf{F}^{-1}(z) \mathbf{T} \mathbf{A}_1^{-1}(z)\right) \\
&= -nb_1^2 E\left[\xi_1 \mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{s}_1\right] + nb_1^2 E\left[\beta_1 \xi_1^2 \mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{s}_1\right] \\
&\quad - b_1^2 E(\beta_1 \xi_1^2) E\left[\text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T}\right] \\
(3.5) \quad &= f_{n1} + f_{n2} + f_{n3} + f_{n4},
\end{aligned}$$

where

$$\begin{aligned}
f_{n1} &= -nb_1^2 E\left[\eta_1 (\mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{s}_1 - \frac{1}{n} \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T})\right], \\
f_{n2} &= -b_1^2 E\left[\left(\text{tr} \mathbf{A}^{-1}(z) \mathbf{T} - E \text{tr} \mathbf{A}^{-1}(z) \mathbf{T}\right) \mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{s}_1\right] \\
&= \frac{b_1^2}{n} E\left[\left(\text{tr} \mathbf{A}^{-1}(z) \mathbf{T} - E \text{tr} \mathbf{A}^{-1}(z) \mathbf{T}\right) \right. \\
&\quad \left. \times \left(\text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T} - E \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T}\right)\right], \\
f_{n3} &= nb_1^2 E\left[\beta_1 \xi_1^2 \left(\mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{s}_1 - \frac{1}{n} \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T}\right)\right]
\end{aligned}$$

and

$$f_{n4} = b_1^2 \left(E\left[\beta_1 \xi_1^2 \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T}\right] - E(\beta_1 \xi_1^2) E\left[\text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T}\right] \right).$$

We conclude from (2.41) and Lemma 3 that

$$\sqrt{h} |f_{n2}| \leq \frac{M}{nv^{5/2}}$$

and that

$$\sqrt{h} |f_{n3}| \leq \frac{M}{\sqrt{nv^2}},$$

because by (2.50) and Lemma 4

$$E|\mathbf{s}_1^T \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{s}_1 - \frac{1}{n} \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T}|^2 \leq \frac{M}{nv}.$$

It follows from Holder's inequality and (2.41) that

$$\begin{aligned}
|f_{n4}| &\leq M(E|\beta_1|^4 E|\xi_1|^4)^{1/4} (E|\text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T} - E \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T}|^2)^{1/2} \\
&\leq \frac{M}{nv^{5/2}}.
\end{aligned}$$

Therefore from (5.3), (3.3), (3.5) (2.2) and (2.21) we obtain

$$\begin{aligned}
n\sqrt{h}D_n &= b_1^2 E \frac{\sqrt{h}}{n} \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T} \mathbf{A}_1^{-1}(z) \mathbf{T} + f_{n1} + O\left(\frac{1}{\sqrt{nv^{5/2}}}\right) \\
&= -b_1^2 E \frac{\sqrt{h}}{n} \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T} \mathbf{A}_1^{-1}(z) \mathbf{T} - b_1^2 \sqrt{h} \frac{EX_{11}^4 - 3}{n} \\
&\quad \times \sum_{k=1}^p E \left[(\mathbf{T}^{1/2} \mathbf{A}_1^{-1}(z) \mathbf{T}^{1/2})_{kk} (\mathbf{T}^{1/2} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T}^{1/2})_{kk} \right] + O\left(\frac{1}{\sqrt{nv^{5/2}}}\right) \\
(3.6) \quad &= -b_1^2 E \frac{\sqrt{h}}{n} \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T} \mathbf{A}_1^{-1}(z) \mathbf{T} + O\left(\frac{1}{\sqrt{nv^{5/2}}}\right).
\end{aligned}$$

A careful inspection on the argument leading to (2.67) indicates that it also works for $E \frac{1}{n} \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T} \mathbf{A}_1^{-1}(z) \mathbf{T}$ and the main difference is that treating the latter does not need to distinguish between the cases $j < k$ and $j > k$. Thus, applying (2.67) with $\mathbf{H} = \mathbf{F}^{-1}(z)$, $z_1 = z_2 = z$ and replacing $(j-1)/n$ there with one we have

$$\begin{aligned}
(3.7) \quad &\frac{1}{n} E \left[\text{tr} \mathbf{T} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T} \mathbf{A}_1^{-1}(z) \right] = -\frac{1}{n} E \left[\text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T} \right] - \\
&\frac{b_{12}(z)b_{12}(z)}{n^2} E \left(\text{tr} \mathbf{T} \mathbf{A}_1^{-1}(z) \mathbf{T} \mathbf{A}_1^{-1}(z) \right) E \left(\text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T} \right) + O\left(\frac{1}{nv^{5/2}}\right).
\end{aligned}$$

We claim that

$$(3.8) \quad |E \underline{m}_n(z) - \underline{m}_n^0(z)| \leq \frac{M}{nv^{3/2}}$$

so that (2.71) is applicable. To prove (3.8), we first show that

$$(3.9) \quad |E \underline{m}_n(z) - \underline{m}_n^0(z)| \leq \frac{M}{nv^2}.$$

Evidently, (2.27) yields

$$\left| \frac{1}{n} E \left(\text{tr} \mathbf{T} \mathbf{A}_1^{-1}(z) \mathbf{T} \mathbf{A}_1^{-1}(z) \right) \right| \leq \frac{M}{v}.$$

It follows from (2.46) and (3.7) that

$$(3.10) \quad \left| \frac{1}{n} E \left[\text{tr} \mathbf{T} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T} \mathbf{A}_1^{-1}(z) \right] \right| \leq \frac{M}{v}.$$

This, together with (3.6), ensures that

$$(3.11) \quad |\mathbf{D}_n| \leq \frac{M}{nv}.$$

It is proved in [6] that (see (6.38) in [6])

$$|1 - c_n \underline{m}_n^0(z) E \underline{m}_n(z) \int \frac{t^2 dH_n(t)}{(1 + t \underline{m}_n^0(z))(1 + t E \underline{m}_n(z))}| \geq M_3 v.$$

Hence (3.9) follows from the above inequality, (3.11) and (3.2). We then conclude from (1.10), (6.6) and (1.11) that

$$(3.12) \quad |1 - c_n \underline{m}_n^0(z) E \underline{m}_n(z) \int \frac{t^2 dH_n(t)}{(1 + t \underline{m}_n^0(z))(1 + t E \underline{m}_n(z))}| \geq M_2 \sqrt{v},$$

which, along with (3.11) and (3.2), immediately gives (3.8).

We are now in a position to use (2.71) with $z_1 = z_2 = z$ and replacing $(j-1)/n$ there with one so that

$$(3.13) \quad \frac{1}{n} E \left[\text{tr} \mathbf{T} \mathbf{A}_1^{-1}(z) \mathbf{T} \mathbf{A}_1^{-1}(z) \right] = \frac{\frac{c_n}{z} \int \frac{t^2 dH_n(t)}{(1 + t \underline{m}_n^0(z))^2}}{1 - c_n \underline{m}_n^0(z) \underline{m}_n^0(z) \int \frac{t^2 dH_n(t)}{(1 + t \underline{m}_n^0(z))^2}} + O\left(\frac{1}{nv^{5/2}}\right).$$

A direct application of (2.45) and (3.8) yields

$$(3.14) \quad \frac{1}{n} E \left[\text{tr} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T} \mathbf{F}_1^{-1}(z) \mathbf{T} \right] = -\frac{c_n}{z^2} \int \frac{t^2}{(1 + t \underline{m}_n^0(z))^3} + O\left(\frac{1}{nv^{5/2}}\right).$$

It follows from (3.7)-(3.14) and (2.70) that

$$(3.15) \quad \sqrt{h} \frac{1}{n} E \left[\text{tr} \mathbf{T} \mathbf{A}_1^{-1}(z) \mathbf{F}^{-1}(z) \mathbf{T} \mathbf{A}_1^{-1}(z) \right] = \sqrt{h} \frac{\frac{c_n}{z} \int \frac{t^2 dH_n(t)}{(1 + t \underline{m}_n^0(z))^3}}{1 - c_n (\underline{m}_n^0(z))^2 \int \frac{t^2 dH_n(t)}{(1 + t \underline{m}_n^0(z))^2}} + O\left(\frac{1}{nv^{5/2}}\right).$$

We then conclude from (3.2), (3.6), (3.15), (1.10) and (3.12) that

$$(3.16) \quad nh(E \underline{m}_n(z) - \underline{m}_n^0(z)) = h \frac{c_n (\underline{m}_n^0(z))^3 \int \frac{t^2 dH_n(t)}{(1 + t \underline{m}_n^0(z))^3}}{\left(1 - c_n (\underline{m}_n^0(z))^2 \int \frac{t^2 dH_n(t)}{(1 + t \underline{m}_n^0(z))^2}\right)^2} + O\left(\frac{1}{\sqrt{nv^{5/2}}}\right).$$

The case when z lies in the vertical lines on the contour can be handled similarly as pointed out in the last section with the truncation version of $\beta_k(z)$ replaced with the truncation version of $n(E \underline{m}_n(z) - \underline{m}_n^0(z))$ (one may refer to [2] as well).

It remains to find the limit of the following

$$(3.17) \quad \frac{1}{4\pi i} \oint K\left(\frac{x-z}{h}\right) \frac{c_n (\underline{m}_n^0(z))^3 \int \frac{t^2 dH_n(t)}{(1 + t \underline{m}_n^0(z))^3}}{\left(1 - c_n (\underline{m}_n^0(z))^2 \int \frac{t^2 dH_n(t)}{(1 + t \underline{m}_n^0(z))^2}\right)^2} dz,$$

which is done in Appendix 3.

4. THE PROOF OF THEOREM 3

For any finite constants l_1, \dots, l_r , by Cauchy's theorem and Fubini's theorem we write

$$\begin{aligned}
 (4.1) \quad & \frac{n}{\sqrt{\ln \frac{1}{h}}} \sum_{j=1}^r l_j \left(F_n(x_j) - \int_{-\infty}^{x_j} \frac{1}{h} \int K\left(\frac{t-y}{h}\right) dF^{c_n, H_n}(y) dt \right) \\
 &= \frac{n}{\sqrt{\ln \frac{1}{h}}} \sum_{j=1}^r l_j \left(\int_{-\infty}^{x_j} f_n(t) dt - \int_{-\infty}^{x_j} \frac{1}{h} \int K\left(\frac{t-y}{h}\right) dF^{c_n, H_n}(y) dt \right) \\
 &= -\frac{n}{2h\pi i \sqrt{\ln \frac{1}{h}}} \sum_{j=1}^r l_j \left(\int_{-\infty}^{x_j} \oint_{\mathcal{C}_1} K\left(\frac{t-z}{h}\right) (tr \mathbf{A}^{-1}(z) - ns_n(z)) dz dt \right) \\
 &= -\frac{n}{2h\pi i \sqrt{\ln \frac{1}{h}}} \sum_{j=1}^r l_j \oint_{\mathcal{C}_1} \left[\int_{-\infty}^{x_j} K\left(\frac{t-z}{h}\right) dt \right] (tr \mathbf{A}^{-1}(z) - ns_n(z)) dz,
 \end{aligned}$$

where the contour \mathcal{C}_1 is defined as before.

Furthermore, we conclude from (2.8) and integration by parts that

$$\begin{aligned}
 & \frac{1}{2h\pi i \sqrt{\ln \frac{1}{h}}} \oint_{\mathcal{C}_1} \left[\int_{-\infty}^x K\left(\frac{t-z}{h}\right) dt \right] (tr \mathbf{A}^{-1}(z) - E tr \mathbf{A}^{-1}(z)) dz \\
 &= -\frac{1}{2h\pi i \sqrt{\ln \frac{1}{h}}} \sum_{k=1}^n (E_k - E_{k-1}) \oint_{\mathcal{C}_1} \left[\int_{-\infty}^x K\left(\frac{t-z}{h}\right) dt \right] \left[\log \beta_k(z) \right]' dz \\
 (4.2) \quad &= \frac{1}{2h\pi i \sqrt{\ln \frac{1}{h}}} \sum_{k=1}^n (E_k - E_{k-1}) \oint_{\mathcal{C}_1} K\left(\frac{x-z}{h}\right) \log \frac{\beta_k^{tr}(z)}{\beta_k(z)} dz,
 \end{aligned}$$

where in the last step one uses the fact that via (1.5)

$$(4.3) \quad \left[\int_{-\infty}^x K\left(\frac{t-z}{h}\right) dt \right]' = K\left(\frac{x-z}{h}\right).$$

It is observed that the unique difference between (4.2) and (2.9) is that the test function $K'(\frac{x-z}{h})$ there is replaced by $K(\frac{x-z}{h})$. Therefore, repeating the arguments in Section 2 we obtain that (4.2) is asymptotically normal with covariance (see (2.82))

$$(4.4) \quad -\frac{1}{2h^2\pi^2 \ln \frac{1}{h}} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} K\left(\frac{x_1-z_1}{h}\right) K\left(\frac{x_2-z_2}{h}\right) \log(\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)) dz_1 dz_2.$$

Also, for the nonrandom part we have

$$(4.5) \quad \frac{1}{2h\pi i \sqrt{\ln \frac{1}{h}}} \oint_{C_1} \left[\int_{-\infty}^x K\left(\frac{t-z}{h}\right) dt \right] n(Etr \mathbf{A}^{-1}(z) - m_n^0(z)) dz.$$

Note that

$$\left| \frac{1}{h} \int_{-\infty}^x K\left(\frac{t-z}{h}\right) dt \right| < \infty.$$

Thus, repeating the arguments in Section 3 we see that (4.5) becomes

$$(4.6) \quad \frac{1}{4h\pi i \sqrt{\ln \frac{1}{h}}} \oint \left[\int_{-\infty}^x K\left(\frac{t-z}{h}\right) dt \right] \frac{c_n(\underline{m}_n^0(z))^3 \int \frac{t^2 dH_n(t)}{(1+t\underline{m}_n^0(z))^3}}{\left(1 - c_n(\underline{m}_n^0(z))^2 \int \frac{t^2 dH_n(t)}{(1+t\underline{m}_n^0(z))^2}\right)^2} dz \\ + O\left(\frac{1}{nh^3 \sqrt{\ln \frac{1}{h}}}\right).$$

The limits of (4.4) and (4.6) are derived in Appendix 3.

5. APPENDIX 1

This Appendix collects some frequently used Lemmas.

Lemma 3. *When z lies in the segments $\gamma_1 \cup \gamma_2$,*

$$|\underline{m}_n^0(z)| \leq M, |Em_n(z)| \leq M, |b_1(z)| \leq M, E|\beta_1(z)|^4 \leq M, E|\beta_1^{tr}(z)|^4 \leq M$$

and

$$(5.1) \quad \frac{1}{n^8} E|tr \mathbf{A}^{-1}(z) \mathbf{D} - Etr \mathbf{A}^{-1}(z) \mathbf{D}|^8 \leq \frac{M}{n^8 v^{12} \|\mathbf{D}\|^8},$$

$$(5.2) \quad E|\eta_1(z)|^8 \leq \frac{M}{n^4 v^4}, \quad E|\xi_1(z)|^8 \leq \frac{M}{n^4 v^4},$$

where \mathbf{D} is a non-random matrix with nonzero spectral norm.

Remark 5. *Lemma 2 in Section 2 improves (5.1) when $\|\mathbf{D}\|$ is not bounded above by a constant but $\frac{1}{n} tr \mathbf{D} \mathbf{D}^* \leq M$.*

Proof. We remind readers that $z = u + iv$ with $v = Mh$ and $u \in [a, b]$ when z lies in the segments $\gamma_1 \cup \gamma_2$.

As pointed out in (6.1) in [6], we obtain

$$(5.3) \quad |\underline{m}_n^0(z)| \leq M, \quad |m_n^0(z)| \leq M.$$

From integration by parts and Theorem 3 in [6] we have for

$$\begin{aligned} |Em_n(z) - m_n^0(z)| &= \left| \int_{-\infty}^{+\infty} \frac{1}{x-z} d(EF^{\mathbf{A}_n}(x) - F^{c_n, H_n}(x)) \right| \\ &= \left| \int_{-\infty}^{+\infty} \frac{EF^{\mathbf{A}_n}(x) - F^{c_n, H_n}(x)}{(x-z)^2} dx \right| \leq \frac{\pi \sup_x |EF^{\mathbf{A}_n}(x) - F^{c_n, H_n}(x)|}{v} \leq M. \end{aligned}$$

This implies

$$(5.4) \quad |Em_n(z)| \leq M, \quad |E\bar{m}_n(z)| \leq M.$$

It follows from Lemma 7 and lemma 8 in [6] that

$$|b_1(z)| \leq M.$$

Repeating the argument of Lemma 3 in [6] gives (5.1).

Write

$$(5.5) \quad \beta_1^{tr}(z) = b_1(z) - \frac{1}{n} \beta_1^{tr}(z) b_1(z) (tr \mathbf{A}^{-1}(z) \mathbf{T} - Etr \mathbf{A}^{-1}(z) \mathbf{T}).$$

We then conclude that

$$\begin{aligned} E|\beta_1^{tr}(z)|^4 &\leq M + \frac{M}{n^4 v^4} E|tr \mathbf{A}^{-1}(z) \mathbf{T} - Etr \mathbf{A}^{-1}(z) \mathbf{T}|^4 \\ (5.6) \quad &\leq M + \frac{M}{n^4 v^{10}} \leq M. \end{aligned}$$

Expand $\beta_1(z)$ as

$$\beta_1(z) = \beta_1^{tr}(z) - \beta_1^{tr}(z) \beta_1(z) \eta_1(z).$$

It follows from (5.6), (2.7) and Lemma 4 that

$$\begin{aligned} E|\beta_1(z)|^4 &\leq E|\beta_1^{tr}(z)|^4 + \frac{1}{v^2} \left(E|\beta_1(z)|^4 E|\eta_1(z) \beta_1^{tr}(z)|^8 \right)^{1/2} \\ &\leq M + \frac{1}{n^2 v^4} \left(E|\beta_1(z)|^4 E|\beta_1^{tr}(z)|^4 \right)^{1/2} \leq M + \frac{1}{n^2 v^4} \left(E|\beta_1(z)|^4 \right)^{1/2}. \end{aligned}$$

Solving the inequality gives

$$E|\beta_1(z)|^4 \leq M.$$

It follows from (2.25) and (5.4) that

$$(5.7) \quad \left| \frac{1}{n} Etr \mathbf{A}_1^{-1}(z) \right| \leq M.$$

By Lemma 4 and (5.1)

$$E|\eta_1(z)|^8 \leq \frac{M}{n^8} E(tr \mathbf{A}_1^{-1}(z) \mathbf{T} \mathbf{A}_1^{-1}(\bar{z}) \mathbf{T})^4 \leq \frac{M}{n^4} E(tr \mathbf{A}_1^{-1}(z) \mathbf{A}_1^{-1}(\bar{z}))^4$$

$$(5.8) \leq \frac{M}{n^4 v^4} E \left[\Im \left(\text{tr} \mathbf{A}_1^{-1}(z) - E \text{tr} \mathbf{A}_1^{-1}(z) \right) \right]^4 + \frac{M}{n^4 v^4} (\Im E \text{tr} \mathbf{A}_1^{-1}(z))^4 \leq \frac{M}{n^4 v^4},$$

where $\mathbf{A}^{-1}(\bar{z})$ denotes the complex conjugate of $\mathbf{A}^{-1}(z)$ and we also use

$$\frac{1}{n} E \text{tr} \mathbf{A}_1^{-1}(z) \mathbf{A}_1^{-1}(\bar{z}) = \frac{1}{v} \Im \left(\frac{1}{n} E \text{tr} \mathbf{A}_1^{-1}(z) \right).$$

This, together with (5.1), yields the estimate of $\xi_1(z)$. \square

Lemma 4. (Lemma 2.7 of [1]) Suppose that X_1, \dots, X_n are i.i.d real random variables with $EX_1 = 0$ and $EX_1^2 = 1$. Let $\mathbf{x} = (X_1, \dots, X_n)^T$ and \mathbf{D} be any $n \times n$ complex matrix. Then for any $p \geq 2$

$$E|\mathbf{x}^T \mathbf{D} \mathbf{x} - \text{tr} \mathbf{D}|^p \leq M_p \left[(E|X_1|^4 \text{tr} \mathbf{D} \mathbf{D}^*)^{p/2} + E|X_1|^{2p} \text{tr}(\mathbf{D} \mathbf{D}^*)^{p/2} \right].$$

6. APPENDIX 2

This section is to verify Remark 1 and Theorem 1.

We first prove that (1.10) is true when \mathbf{T} becomes the identity matrix. When \mathbf{T} is the identity matrix, the left hand of (1.10) becomes

$$1 - c_n \frac{(\underline{m}_n^0(z))^2}{(1 + \underline{m}_n^0(z))^2}.$$

In view of (1.1) we have

$$(6.1) \quad 1 - c_n \frac{(\underline{m}_n^0(z))^2}{(1 + \underline{m}_n^0(z))^2} = 1 - \frac{1}{c_n} (1 + z \underline{m}_n^0(z))^2$$

and

$$(6.2) \quad \underline{m}_n^0(z) = \frac{-(z + 1 - c_n) + \sqrt{(z - 1 - c_n)^2 - 4c_n}}{2z}.$$

Thus,

$$\begin{aligned} 1 - c_n \frac{(\underline{m}_n^0(z))^2}{(1 + \underline{m}_n^0(z))^2} &= 1 - \frac{1}{c_n} \left[\frac{-(z - 1 - c_n) + \sqrt{(z - 1 - c_n)^2 - 4c_n}}{2} \right]^2 \\ &= \frac{1}{2c_n} \frac{\sqrt{4c_n - (z - 1 - c_n)^2}}{2c} \left[(z - 1 - c_n)i + \sqrt{4c_n - (z - 1 - c_n)^2} \right] \\ &= \frac{\sqrt{(z - 1 - c_n)^2 - 4c_n}}{2c_n} \left[(z - 1 - c_n) + \sqrt{(z - 1 - c_n)^2 - 4c_n} \right] \\ &= \frac{\sqrt{(z - 1 - c_n)^2 - 4c_n}}{c_n} (c_n z \underline{m}_n^0(z) + z - 1) \\ &= \frac{\sqrt{(z - 1 - c_n)^2 - 4c_n}}{2c_n} \frac{(1 + c_n \underline{m}_n^0(z))}{c_n \underline{m}_n^0(z)}, \end{aligned}$$

where in the last two steps one uses the facts that

$$(6.3) \quad m_n^0(z) = -\frac{c_n + z - 1 - \sqrt{(z - 1 - c_n)^2 - 4c_n}}{c_n z}$$

and

$$(6.4) \quad m_n^0(z) = \frac{1}{1 - c_n - c_n z m_n^0(z) - z}.$$

It follows from (6.4) and (5.3) that

$$(6.5) \quad \left| \frac{1}{1 + c_n m_n^0(z)} \right| = |1 - c_n - z c_n m_n^0(z)| \leq M.$$

Write

$$\sqrt{(z - 1 - c_n)^2 - 4c_n} = \sqrt{(a - z)(b - z)}.$$

Then it is simple to verify that

$$|\sqrt{(a - z)(b - z)}| \geq \sqrt{(b - a)v}.$$

Thus, (1.10) is true when \mathbf{T} is the identity matrix.

Lemma 5. *Under the assumptions that $n^5 h^{29/2} \leq M$ and that*

$$(6.6) \quad \int \frac{dH_n(t)}{|1 + t \underline{m}_n^0(z)|^{20}} < \infty,$$

(1.11) is true and

$$(6.7) \quad \int \frac{dH_n(t)}{|z - \frac{n-1}{n} t b_{12}(z)|^4} < \infty.$$

Remark 6. *The assumptions that $n^5 h^{29/2} \leq M$ and (6.6) are only used in this Lemma.*

Proof. First recall that (see (6.30) in [6])

$$(6.8) \quad |E \underline{m}_n(z) - \underline{m}_n^0(z)| \leq \frac{M}{n v^{5/2}}.$$

It is straightforward to verify that

$$(6.9) \quad |1 + t E \underline{m}_n(z)|^2 - |1 + t \underline{m}_n^0(z)|^2 \leq M |E \underline{m}_n(z) - \underline{m}_n^0(z)|.$$

We then write

$$(6.10) \quad \begin{aligned} & \int \frac{dH_n(t)}{|1 + t E \underline{m}_n(z)|^4} - \int \frac{dH_n(t)}{|1 + t \underline{m}_n^0(z)|^4} \\ &= - \int \frac{|1 + t E \underline{m}_n(z)|^2 - |1 + t \underline{m}_n^0(z)|^2 dH_n(t)}{|1 + t E \underline{m}_n(z)|^4 |1 + t \underline{m}_n^0(z)|^2} \\ & \quad - \int \frac{|1 + t E \underline{m}_n(z)|^2 - |1 + t \underline{m}_n^0(z)|^2 dH_n(t)}{|1 + t E \underline{m}_n(z)|^2 |1 + t \underline{m}_n^0(z)|^4}. \end{aligned}$$

Obviously, for the above last term, by Holder's inequality, (6.6), (6.8) and (6.9)

$$\begin{aligned}
 & \left| \int \frac{|1 + tE\underline{m}_n(z)|^2 - |1 + t\underline{m}_n^0(z)|^2 dH_n(t)}{|1 + tE\underline{m}_n(z)|^2 |1 + t\underline{m}_n^0(z)|^4} \right| \\
 & \leq M |E\underline{m}_n(z) - \underline{m}_n^0(z)| \left(\int \frac{dH_n(t)}{|1 + tE\underline{m}_n(z)|^4} \int \frac{dH_n(t)}{|1 + t\underline{m}_n^0(z)|^8} \right)^{1/2} \\
 (6.11) \quad & \leq \frac{M}{nv^{5/2}} \left(\int \frac{dH_n(t)}{|1 + tE\underline{m}_n(z)|^4} \right)^{1/2}.
 \end{aligned}$$

As for another term in (6.10), using (6.9) successively we have

$$\begin{aligned}
 & \left| \int \frac{|1 + tE\underline{m}_n(z)|^2 - |1 + t\underline{m}_n^0(z)|^2 dH_n(t)}{|1 + tE\underline{m}_n(z)|^4 |1 + t\underline{m}_n^0(z)|^2} \right| \\
 & \leq \int \frac{M |E\underline{m}_n(z) - \underline{m}_n^0(z)| dH_n(t)}{|1 + tE\underline{m}_n(z)|^4 |1 + t\underline{m}_n^0(z)|^2} \\
 & \leq \int \frac{M |E\underline{m}_n(z) - \underline{m}_n^0(z)| dH_n(t)}{|1 + tE\underline{m}_n(z)|^2 |1 + t\underline{m}_n^0(z)|^4} + \int \frac{M |E\underline{m}_n(z) - \underline{m}_n^0(z)|^2 dH_n(t)}{|1 + tE\underline{m}_n(z)|^4 |1 + t\underline{m}_n^0(z)|^4} \\
 & \leq \frac{M}{nv^{5/2}} \left(\int \frac{dH_n(t)}{|1 + tE\underline{m}_n(z)|^4} \right)^{1/2} + \frac{M}{(nv^{5/2})^2} \int \frac{dH_n(t)}{|1 + tE\underline{m}_n(z)|^4 |1 + t\underline{m}_n^0(z)|^4} \\
 & \leq \dots \\
 & \leq \left(\frac{M}{nv^{5/2}} + \frac{M}{(nv^{5/2})^2} + \frac{M}{(nv^{5/2})^3} + \frac{M}{(nv^{5/2})^4} \right) \left(\int \frac{dH_n(t)}{|1 + tE\underline{m}_n(z)|^4} \right)^{1/2} \\
 & \quad + \frac{M}{(nv^{5/2})^5} \left(\int \frac{dH_n(t)}{|1 + tE\underline{m}_n(z)|^4 |1 + t\underline{m}_n^0(z)|^{10}} \right)^{1/2} \\
 & \leq \frac{M}{nv^{5/2}} \left(\int \frac{dH_n(t)}{|1 + tE\underline{m}_n(z)|^4} \right)^{1/2} + \frac{M}{(nv^{5/2})^5 v^2} \left(\int \frac{dH_n(t)}{|1 + tE\underline{m}_n(z)|^4} \right)^{1/2}.
 \end{aligned}$$

where in the last step one uses (6.16) in [6]. This, together with (6.11) and (6.10), yields

$$\left| \int \frac{dH_n(t)}{|1 + tE\underline{m}_n(z)|^4} \right| \leq M + \left[\frac{M}{nv^{5/2}} + \frac{M}{(nv^{5/2})^5 v^2} \right] \left(\int \frac{dH_n(t)}{|1 + tE\underline{m}_n(z)|^4} \right)^{1/2}.$$

Solving the inequality gives

$$\int \frac{dH_n(t)}{|1 + tE\underline{m}_n(z)|^4} < \infty.$$

Consider (6.7) now. By (2.69), (2.25), (6.8) and Lemma 3

$$|b_{12}(z) + z\underline{m}_n^0(z)| \leq \frac{M}{nv^{5/2}}.$$

Applying this inequality and repeating the argument for (1.11) we may prove (6.7) and omit the details here.

Proof of Theorem 1. Write

$$\begin{aligned} & nh \left[\frac{1}{h} \int_a^b K\left(\frac{x-y}{h}\right) dF^{c_n, H_n}(y) - f_{c_n, H_n}(x) \right] \\ &= nh \left[\int_{\frac{x-b}{h}}^{\frac{x-a}{h}} K(y) f_{c_n, H_n}(x-yh) dy - f_{c_n, H_n}(x) \right]. \end{aligned}$$

By Taylor's expansion

$$f_{c_n, H_n}(x-yh) = f_{c_n, H_n}(x) - f'_{c_n, H_n}(x)yh + f''_{c_n, H_n}(x_0)(yh)^2,$$

where x_0 lies in $[x-yh, x]$. This, together with (1.7), (1.15), (1.14) and Theorem 2, ensures Theorem 1.

7. APPENDIX 3

The aim in this section is to develop the asymptotic means and variances in Theorem 2 and Theorem 3. Consider (2.82) first. Note that

$$\begin{aligned} (2.82) &= -\frac{1}{2h^2\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} K'\left(\frac{x_1-z_1}{h}\right) K'\left(\frac{x_2-z_2}{h}\right) \\ &\quad \times \left[\ln \left| \underline{m}_n^0(z_1) - \underline{m}_n^0(z_2) \right| + i \arg(\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)) \right] dz_1 dz_2, \end{aligned}$$

where the contours \mathcal{C}_1 and \mathcal{C}_2 are two rectangles defined in (2.3) and (2.81), respectively.

As in Section 5 of [2] one may prove that

$$(7.1) \quad \inf_{z \in S, n} |\underline{m}_n^0(z)| > 0, \quad \left| \frac{\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)}{z_1 - z_2} \right| \geq \frac{1}{2} |\underline{m}_n^0(z_1) \underline{m}_n^0(z_1)|,$$

where S is any bounded subset of \mathbb{C} .

To facilitate statements, denote the real parts of z_j by $u_j, j = 1, 2$. In what follows, let $n \rightarrow \infty$ first and then $v_0 \rightarrow 0$. Then, as argued in [2], the integrals in (7.1) involving the arg term and the vertical sides approach zero.

Define

$$\begin{aligned} K_{ri}^{(1)} &= K'_r\left(\frac{x_1-z_1}{h}\right) K'_r\left(\frac{x_2-z_2}{h}\right) - K'_i\left(\frac{x_1-z_1}{h}\right) K'_i\left(\frac{x_2-z_2}{h}\right), \\ K_{ri}^{(2)} &= K'_r\left(\frac{x_1-z_1}{h}\right) K'_r\left(\frac{x_2-z_2}{h}\right) + K'_i\left(\frac{x_1-z_1}{h}\right) K'_i\left(\frac{x_2-z_2}{h}\right). \end{aligned}$$

Therefore it is enough to investigate the following integrals

$$-\frac{1}{h^2\pi^2} \int_{a_l}^{a_r} \int_{a_l-\varepsilon}^{a_r+\varepsilon} [K_{ri}^{(1)} \ln |\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)| - K_{ri}^{(2)} \ln |\underline{m}_n^0(z_1) - \overline{\underline{m}_n^0}(z_2)|] du_1 du_2$$

$$(7.2) = \frac{1}{h^2\pi^2} \int_{a_l}^{a_r} \int_{a_l-\varepsilon}^{a_r+\varepsilon} (K_r'(\frac{x_1-z_1}{h}) K_r'(\frac{x_2-z_2}{h}) \ln \left| \frac{\underline{m}_n^0(z_1) - \overline{\underline{m}_n^0}(z_2)}{\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)} \right|) du_1 du_2$$

$$(7.3) + \frac{1}{h^2\pi^2} \int_{a_l}^{a_r} \int_{a_l-\varepsilon}^{a_r+\varepsilon} (K_i'(\frac{x_1-z_1}{h}) K_i'(\frac{x_2-z_2}{h}) \times \ln \left| (\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2))(\underline{m}_n^0(z_1) - \overline{\underline{m}_n^0}(z_2)) \right|) du_1 du_2,$$

where $K_r'(\frac{x-z}{h})$ and $K_i'(\frac{x-z}{h})$, respectively, represent the real part and imaginary part of $K'(\frac{x-z}{h})$, $\overline{\underline{m}_n^0}(z)$ stands for the complex conjugate of $\underline{m}_n^0(z)$.

We develop the limit of (7.2) and (7.3) below. To this end, we list some facts below. By (1.5) and (1.6) one may verify that

$$(7.4) \quad \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| K'(u_1) K'(u_2) \ln(u_1 - u_2)^2 \right| du_1 du_2 < \infty.$$

In addition, it follows from (1.5) that

$$\ln \frac{1}{h^2} \int_{\frac{x-b}{h}}^{\frac{x-a}{h}} K_r'(u_1) \int_{\frac{x-b-\varepsilon}{h}}^{\frac{x-a+\varepsilon}{h}} K_r'(u_2) du_1 du_2 \rightarrow 0.$$

This, together with (7.4), implies that as $n \rightarrow \infty$

$$\begin{aligned} & \frac{1}{h^2} \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) \ln(u_1 - u_2)^2 du_1 du_2 \\ &= \int_{\frac{x_1-a_r}{h}}^{\frac{x_1-a_l}{h}} \int_{\frac{x_2-a_r-\varepsilon}{h}}^{\frac{x_2-a_l+\varepsilon}{h}} K'(u_1) K'(u_2) \left[\ln(u_1 - u_2)^2 - \ln \frac{1}{h^2} \right] du_1 du_2 \\ (7.5) \quad & \rightarrow \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K'(u_1) K'(u_2) \ln(u_1 - u_2)^2 du_1 du_2. \end{aligned}$$

By (1.8) and the continuity property of $K''(u + iv_0)$ and $K'(u + iv_0)$ in u and v_0 it is not difficult to prove that

$$(7.6) \quad \lim_{v_0 \rightarrow 0} \int_{-\infty}^{+\infty} |K''(u + iv_0)| du = \int_{-\infty}^{+\infty} |K''(u)| du$$

and

$$(7.7) \quad \lim_{v_0 \rightarrow 0} \int_{-\infty}^{+\infty} K^{(j)}(u + iv_0) du = \int_{-\infty}^{+\infty} K^{(j)}(u) du, \quad j = 0, 1,$$

where $K^{(j)}$ is the j -th derivative of K .

By complex Roller's theorem

$$(7.8) \quad K'_i\left(\frac{x-z_1}{h}\right) = K'_i\left(\frac{x-u_1}{h} + iv_0\right) = vK''_r\left(\frac{x-u}{h} + iv_1\right)$$

because $K'_i\left(\frac{x-u_1}{h}\right) = 0$, where v_1 lies in $(0, v_0)$. Thus we conclude from (7.1) and (7.6) that

$$\begin{aligned} & \left| \frac{1}{h} \int_{a_i}^{a_r} \left(K'_i\left(\frac{x_1-z_1}{h}\right) \ln \left| (\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2))(\underline{m}_n^0(z_1) - \overline{\underline{m}_n^0}(z_2)) \right| \right) du_1 \right| \\ & \leq v_0 h \ln(v_0^{-1}h) \frac{1}{h} \int_a^b |K''\left(\frac{x-u}{h} + iv_1\right)| du_1 \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, $v_0 \rightarrow 0$, which implies that (7.3) converges to zero.

Consider (7.2) next. We claim that for $u \in [\frac{x-b}{h}, \frac{x-a}{h}]$, as $n \rightarrow \infty$,

$$(7.9) \quad |\underline{m}_n^0(z_n) - \underline{m}(u_n)| \rightarrow 0,$$

where $z_n = u_n - iv_0h$ with $u_n = x - uh$. Indeed, from (3.10) in [1] we have

$$(7.10) \quad z(\underline{m}_n^0) = -\frac{1}{\underline{m}_n^0} + c_n \int \frac{tdH_n(t)}{1 + t\underline{m}_n^0},$$

(one may also refer to Section 6.3 of [6]). Then, as pointed out in Lemma 1 [6], relying on this expression we may draw the conclusions for \underline{m}_n^0 similar to those in Theorem 1.1 of [13] for $\underline{m}(z)$. Thus we have

$$(7.11) \quad |\underline{m}_n^0(z_n) - \underline{m}_n^0(u_n)| \rightarrow 0.$$

Also, the argument of Lemma 2 in [6] gives

$$(7.12) \quad |\underline{m}_n^0(u_n) - \underline{m}(u_n)| \rightarrow 0.$$

Therefore, (7.9) is true, as claimed.

Now, as in [2], for (7.2) write

$$(7.13) \quad \ln \left| \frac{\underline{m}_n^0(z_1) - \overline{\underline{m}_n^0}(z_2)}{\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)} \right| = \frac{1}{2} \ln \left(1 + \frac{4\underline{m}_{ni}^0(z_1)\underline{m}_{ni}^0(z_2)}{|\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)|^2} \right),$$

where $\underline{m}_{ni}^0(z)$ denotes the imaginary part of $\underline{m}_n^0(z)$. By (7.1)

$$(7.14) \quad \ln \left(1 + \frac{4\underline{m}_{ni}^0(z_1)\underline{m}_{ni}^0(z_2)}{|\underline{m}_n^0(z_1) - \underline{m}_n^0(z_2)|^2} \right) \leq \ln \left(1 + \frac{16\underline{m}_{ni}^0(z_1)\underline{m}_{ni}^0(z_2)}{(u_1 - u_2)^2 |\underline{m}_n^0(z_1)\underline{m}_n^0(z_2)|^2} \right).$$

In view of (7.1) and Lemma 3

$$(7.15) \quad \sup_{u_1, u_2 \in [a, b], v_1, v_2 \in [v_0 h, 1]} \left| \frac{\underline{m}_{ni}^0(z_1)\underline{m}_{ni}^0(z_2)}{|\underline{m}_n^0(z_1)\underline{m}_n^0(z_2)|^2} \right| < \infty.$$

By the generalized dominated convergence theorem we then conclude from (7.5), (7.7), (7.9), (7.14), (7.15) that as $n \rightarrow \infty$

$$\int_{\frac{x_1-a_r}{h}}^{\frac{x_1-a_l}{h}} \int_{\frac{x_2-a_r-\varepsilon}{h}}^{\frac{x_2-a_l+\varepsilon}{h}} K'_r(z_1) K'_r(z_2) \left[\ln \left| \frac{\underline{m}_n^0(u_{n1} - iv_0 h) - \overline{m}_n^0(u_{n2} - iv_0 h/2)}{\underline{m}_n^0(u_{n1} - iv_0 h) - \underline{m}_n^0(u_{n2} - iv_0 h/2)} \right| \right. \\ \left. - \ln \left| \frac{\underline{m}(u_{n1}) - \overline{m}(u_{n2})}{\underline{m}(u_{n1}) - \underline{m}(u_{n2})} \right| \right] du_1 du_2 \rightarrow 0,$$

where $u_{nj} = x_j - u_j h$, $j = 1, 2$. In addition, it follows from (7.5), (7.7), and inequalities similar to (7.14) and (7.15) that as $n \rightarrow \infty$ and then $v_0 \rightarrow 0$

$$\int_{\frac{x_1-a_r}{h}}^{\frac{x_1-a_l}{h}} \int_{\frac{x_2-a_r-\varepsilon}{h}}^{\frac{x_2-a_l+\varepsilon}{h}} (K'_r(z_1) K'_r(z_2) - K'_r(u_1) K'_r(u_2)) \ln \left| \frac{\underline{m}(u_{n1}) - \overline{m}(u_{n2})}{\underline{m}(u_{n1}) - \underline{m}(u_{n2})} \right| du_1 du_2 \rightarrow 0.$$

Therefore (7.2) can be reduced to the following

$$(7.16) \quad \int_{\frac{x_1-a_r}{h}}^{\frac{x_1-a_l}{h}} \int_{\frac{x_2-a_r-\varepsilon}{h}}^{\frac{x_2-a_l+\varepsilon}{h}} K'(u_1) K'(u_2) \ln \left| \frac{\underline{m}(u_{n1}) - \overline{m}(u_{n2})}{\underline{m}(u_{n1}) - \underline{m}(u_{n2})} \right| du_1 du_2 + o(1),$$

which turns to be

$$\frac{1}{h^2} \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} K'\left(\frac{u_1}{h}\right) K'\left(\frac{u_2}{h}\right) \ln \left| \frac{\underline{m}(x_1 - u_1) - \overline{m}(x_2 - u_2)}{\underline{m}(x_1 - u_1) - \underline{m}(x_2 - u_2)} \right| du_1 du_2 + o(1).$$

To handle (7.16), we need two more lemmas:

Lemma 6. Suppose that the function $g(x_1, x_2)$ is continuous in x_1 and x_2 ,

$$(7.17) \quad \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r}^{x_2-a_l} |g(x_1 - u_1, x_2 - u_2)| du_1 du_2 < \infty$$

and

$$(7.18) \quad \int_{x_1-a_r}^{x_1-a_l} |g(x_1 - u_1, x_2)| du_1 < \infty, \quad \int_{x_2-a_r}^{x_2-a_l} |g(x_1, x_2 - u_2)| du_2 < \infty.$$

Then, as $n \rightarrow \infty$

$$(7.19) \quad \frac{1}{h^2} \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} K'\left(\frac{u_1}{h}\right) K'\left(\frac{u_2}{h}\right) g(x_1 - u_1, x_2 - u_2) du_1 du_2 \rightarrow 0,$$

where $x_1 \neq a_l, a_r$ and $x_2 \neq a_l, a_r$.

Define the sets $G_1 = (|u_1| \leq \delta_1) \cap (|u_2| > \delta_2)$, $G_2 = (|u_1| > \delta_1) \cap (|u_2| \leq \delta_2)$ and $G_3 = (|u_1| > \delta_1) \cap (|u_2| > \delta_2)$. Splitting the region of integration into the union of the sets $(|u_1| \leq \delta_1) \cap (|u_2| \leq \delta_2)$, G_1 , G_2 and G_3 gives

$$\left| \frac{1}{h^2} \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} K'\left(\frac{u_1}{h}\right) K'\left(\frac{u_2}{h}\right) [g(x_1 - u_1, x_2 - u_2) - g(x_1, x_2)] du_1 du_2 \right|$$

$$(7.20) \quad \leq I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$\begin{aligned} I_1 &= \sup_{|u_1| \leq \delta_1, |u_2| \leq \delta_2} \left| g(x_1 - u_1, x_2 - u_2) - g(x_1, x_2) \right| \int_{-\infty}^{+\infty} |K'(u)| du \Big|^2, \\ I_2 &= |g(x_1, x_2)| \left| \frac{1}{h^2} \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} I(G_1 \cup G_2 \cup G_3) K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) du_1 du_2 \right|, \\ I_3 &= \left| \frac{1}{h^2} \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} I(G_1) K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) g(x_1 - u_1, x_2 - u_2) du_1 du_2 \right|, \\ I_4 &= \left| \frac{1}{h^2} \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} I(G_2) K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) g(x_1 - u_1, x_2 - u_2) du_1 du_2 \right| \end{aligned}$$

and

$$I_5 = \left| \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} I(G_3) \frac{u_1 u_2}{h^2} K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) \frac{g(x_1 - u_1, x_2 - u_2)}{u_1 u_2} du_1 du_2 \right|.$$

Evidently, $I_1 \rightarrow 0$ due to the continuity property of $g(x_1, x_2)$ when δ_1 and δ_2 converge to zero. As $n \rightarrow \infty$, for I_2 we have

$$I_2 \leq M |g(x_1, x_2)| \int_{|u| > \delta/h} |K'(u)| du \int_{-\infty}^{+\infty} |K'(u)| du \rightarrow 0,$$

and for I_5 by (7.18) we obtain

$$\begin{aligned} I_5 &\leq \frac{1}{\delta_1 \delta_2} \sup_{|u_1| > \delta_1/h} |u_1 K'(u_1)| \sup_{|u_2| > \delta_2/h} |u_2 K'(u_2)| \\ &\quad \times \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} |g(x_1 - u_1, x_2 - u_2)| du_1 du_2 \rightarrow 0. \end{aligned}$$

Consider I_3 . Similar to I_5 ,

$$I_3 \leq \frac{1}{\delta_2} \sup_{|u_2| > \delta_2/h} |u_2 K'(u_2)| \int_{|u_1| \leq \delta_1/h} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} |K'(u_1) g(x_1 - u_1 h, x_2 - u_2)| du_1 du_2.$$

While, as $n \rightarrow \infty$ and then $\delta_1 \rightarrow 0$, by the dominated convergence theorem

$$\frac{1}{h} \int_{|u_1| \leq \delta_1} |K'(\frac{u_1}{h})| \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} |(g(x_1 - u_1, x_2 - u_2) - g(x_1, x_2 - u_2))| du_1 du_2 \rightarrow 0.$$

From (7.18) we then see that $I_3 \rightarrow 0$. One may similarly prove that I_4 converges to zero as well. We summarize the above that (7.24) converges to zero as

$n \rightarrow \infty$ first and then both $\delta_1 \rightarrow 0$ and $\delta_2 \rightarrow 0$. In addition, apparently,

$$\begin{aligned}
 (7.21) \quad g(x_1, x_2) \frac{1}{h} \int_{x_1 - a_r}^{x_1 - a_l} K'(\frac{u}{h}) du &= g(x_1, x_2) \int_{\frac{x_1 - a_l}{h}}^{\frac{x_1 - a_r}{h}} K'(u) du \\
 &= g(x_1, x_2) K(u) \Big|_{\frac{x_1 - a_l}{h}}^{\frac{x_1 - a_r}{h}} \rightarrow 0.
 \end{aligned}$$

Thus (7.19) is proved.

The next lemma extends (1.6) in [13], which now includes the boundary points of $F^{c,H}(x)$ under some extra conditions.

Lemma 7. *Suppose that the support of $F^{c,H}(x)$ is $[a, b]$ with $a > 0$ and b finite. Then $\underline{m}(x)$ is the unique solution to the equation*

$$(7.22) \quad x = -\frac{1}{\underline{m}(x)} + c \int \frac{\lambda dH(\lambda)}{1 + \lambda \underline{m}(x)},$$

where $\lim_{z \rightarrow x} \underline{m}(z) = \underline{m}(x)$.

Proof. When u , the real part of z , is bounded, we have

$$\Im(\underline{m}(z)) \geq \frac{v}{M + v^2}.$$

It follows that

$$(7.23) \quad \frac{v}{\Im(\underline{m}(z))} \leq M + v^2.$$

Considering the imaginary parts of both sides of the equality (1.3) yields

$$v = \frac{\Im(\underline{m}(z))}{|\underline{m}(z)|^2} - c \Im(\underline{m}(z)) \int \frac{t^2 dH(t)}{|1 + t \underline{m}(z)|^2},$$

which, together with (7.1) and (7.23), implies

$$(7.24) \quad \sup_{z \in S} \int \frac{t^2 dH(t)}{|1 + t \underline{m}(z)|^2} \leq M.$$

Taking $z \rightarrow x$ in (1.3) and using (1.5) in [13] we then see that (7.22) is true. The uniqueness of $\underline{m}(x)$ is from continuity of $\underline{m}(x)$ and the uniqueness of $\underline{m}(x)$ when $\Im \underline{m}(x) \neq 0$ given in [13]. \square

We are now in a position to apply Lemma 6 to (7.16). It follows from Lemma 7 that $\underline{m}(x_1) \neq \underline{m}(x_2)$ and $\underline{m}(x_1) \neq \overline{m}(x_2)$ whenever $x_1 \neq x_2$. Also, note (5.1) in [2]. Therefore $g(x_1, x_2) = \ln \left| \frac{\underline{m}(x_1) - \overline{m}(x_2)}{\underline{m}(x_1) - \underline{m}(x_2)} \right|$ is continuous in x_1 and x_2 . Furthermore, it is straightforward to show that $\ln \left(1 + \frac{M}{(x_1 - x_2) - (u_1 - u_2)} \right)$ for $u_1, u_2 \in [a_l - \varepsilon, a_r + \varepsilon]$ is Lebesgue integrable and $\ln \left(1 + \frac{M}{(x_1 - x_2) - (u_1)} \right)$ for

$u_2 \in [a_l - \varepsilon, a_r + \varepsilon]$ is Lebesgue integrable. Thus, in view of inequalities similar to (7.13)-(7.15) and applying (7.19) we have

$$(7.25) \quad \frac{1}{h^2} \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) \ln \left| \frac{\underline{m}(x_1 - u_1) - \overline{m}(x_2 - u_2)}{\underline{m}(x_1 - u_1) - \underline{m}(x_2 - u_2)} \right| du_1 du_2 \rightarrow 0,$$

which is the limit of (7.2) due to (7.16) when $x_1 \neq x_2$.

When $x_1 = x_2 = x$ taking $g(x_1, x_2) = \ln \left| \underline{m}(x) - \overline{m}(x) \right|$ and applying (7.19) we obtain

$$(7.26) \quad \frac{1}{h^2} \int_{x - a_r}^{x - a_l} \int_{x - a_r - \varepsilon}^{x - a_l + \varepsilon} K'(\frac{u_1}{h}) K'(\frac{u_2}{h}) \ln \left| \underline{m}(x - u_1) - \overline{m}(x - u_2) \right| du_1 du_2 \rightarrow 0.$$

Here we keep in mind that the boundary points are not considered when investigating the case $x_1 = x_2 = x$. Consider next

$$(7.27) \quad \frac{1}{h^2} \int_{x-b}^{x-a} K'(\frac{u_1}{h}) \int_{x-b-\varepsilon}^{x-a+\varepsilon} K'(\frac{u_2}{h}) \ln \left| \underline{m}(x - u_1) - \underline{m}(x - u_2) \right| du_1 du_2.$$

By complex Roller's theorem we have

$$(7.28) \quad \begin{aligned} & \ln \left| \underline{m}(x - u_1) - \underline{m}(x - u_2) \right| \\ &= \frac{1}{2} \ln \left((u_1 - u_2)^2 [|\underline{m}'_r(x - u_3)|^2 + |\underline{m}'_i(x - u_4)|^2] \right) \\ &= \frac{1}{2} \ln(u_1 - u_2)^2 + \frac{1}{2} g_{ri}(x - u_1, x - u_2), \end{aligned}$$

where $g_r(x - u_1, x - u_2) = \ln \left(|\underline{m}'_r(t_1(x - u_1) + (1 - t_1)(x - u_2))|^2 + |\underline{m}'_i(t_2(x - u_1) + (1 - t_2)(x - u_2))|^2 \right)$, $u_3 = t_1 u_1 + (1 - t_1) u_2$, $u_4 = t_2 u_1 + (1 - t_2) u_2$ and $t_1, t_2 \in (0, 1)$. It follows from inequalities for $\underline{m}(x)$ similar to (7.1) that

$$\left| \int_{x-b}^{x-a} \int_{x-b-\varepsilon}^{x-a+\varepsilon} \ln \left| \underline{m}(x - u_1) - \underline{m}(x - u_2) \right| du_1 du_2 \right| < \infty.$$

This, together with (7.28), ensures that

$$\left| \int_{x-b}^{x-a} \int_{x-b-\varepsilon}^{x-a+\varepsilon} g_r(x - u_1, x - u_2) du_1 du_2 \right| < \infty.$$

Similarly, one may verify the remaining conditions in Lemma 6. Therefore, using Lemma 6 with $g(x_1, x_2) = \ln |\underline{m}'(x)|^2$ gives

$$(7.29) \quad \frac{1}{h^2} \int_{x-b}^{x-a} K'(\frac{u_1}{h}) \int_{x-b-\varepsilon}^{x-a+\varepsilon} K'(\frac{u_2}{h}) g_r(x - u_1, x - u_2) du_1 du_2 \rightarrow 0.$$

We then conclude from (7.28), (7.29) and (7.5) that

$$(7.27) = \frac{1}{2} \frac{1}{h^2} \int_{x-b}^{x-a} K'(\frac{u_1}{h}) \int_{x-b-\varepsilon}^{x-a+\varepsilon} K'(\frac{u_2}{h}) \ln(u_1 - u_2)^2 du_1 du_2 + o(1)$$

$$(7.30) \rightarrow \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K'(u_1) K'(u_2) \ln(u_1 - u_2)^2 du_1 du_2.$$

which is the opposite number of the limit of (7.2) due to (7.26) and (7.16) when $x_1 = x_2$.

Limit of (3.17). From an expression similar to (1.3) we obtain

$$\frac{d}{dz} \underline{m}_n^0(z) = \frac{(\underline{m}_n^0(z))^2}{1 - c \int \frac{t^2 (\underline{m}_n^0(z))^2}{(1 + t \underline{m}_n^0(z))^2} dH_n(t)}.$$

It follows that (3.17) becomes

$$(7.31) \quad \frac{1}{4\pi i} \oint K(\frac{x-z}{h}) \frac{d}{dz} \log \left[1 - c \int \frac{t^2 (\underline{m}_n^0(z))^2}{(1 + t \underline{m}_n^0(z))^2} dH_n(t) \right] dz$$

$$= \frac{1}{4\pi h i} \oint K'(\frac{x-z}{h}) \log \left[1 - c \int \frac{t^2 (\underline{m}_n^0(z))^2}{(1 + t \underline{m}_n^0(z))^2} dH_n(t) \right] dz$$

As in the inequality above (6.37) in [6] and (3.21) in [1] one may prove that

$$(7.32) \quad |1 - c \int \frac{t^2 (\underline{m}_n^0(z))^2}{(1 + t \underline{m}_n^0(z))^2} dH_n(t)| \geq Mv.$$

This implies that the integrals on the two vertical lines in (7.31) are bounded by $Mv \log v^{-1}$, which converges to zero as $v \rightarrow 0$. The integrals on the two horizontal lines are equal to

$$(7.33) \quad \frac{1}{2\pi h} \int K'_i(\frac{x-z}{h}) \log \left| 1 - c \int \frac{t^2 (\underline{m}_n^0(z))^2}{(1 + t \underline{m}_n^0(z))^2} dH_n(t) \right| du$$

$$(7.34) \quad + \frac{1}{2\pi h} \int K'_r(\frac{x-z}{h}) \arg \left[1 - c \int \frac{t^2 (\underline{m}_n^0(z))^2}{(1 + t \underline{m}_n^0(z))^2} dH_n(t) \right] du.$$

By (2.19) in [2], (7.8) and (7.32) we see that (7.33) is bounded by $Mv \log v^{-1}$, converging to zero. It follows from (7.9) and Lemma 5 that

$$\int \frac{t^2 (\underline{m}_n^0(z_n))^2}{(1 + t \underline{m}_n^0(z_n))^2} dH_n(t) - \int \frac{t^2 (\underline{m}(u_n))^2}{(1 + t \underline{m}(u_n))^2} dH_n(t) \rightarrow 0.$$

We also claim that

$$(7.35) \quad \int \frac{t^2 (\underline{m}(u_n))^2}{(1 + t \underline{m}(u_n))^2} dH_n(t) - \int \frac{t^2 (\underline{m}(u_n))^2}{(1 + t \underline{m}(u_n))^2} dH(t) \rightarrow 0.$$

To see this, introduce random variables T_n having distribution $H_n(t)$ and T having distribution $H(t)$. Then $T_n \xrightarrow{D} T$. Also T_n and T are both bounded. Consequently by Lemma 5

$$\begin{aligned} & E \left| \frac{T_n^2}{(1 + T_n \underline{m}(u_n))^2} - \frac{T^2}{(1 + T \underline{m}(u_n))^2} \right| \\ & \leq \left(E \left| \frac{T_n}{1 + T_n \underline{m}(u_n)} - \frac{T}{1 + T \underline{m}(u_n)} \right|^2 E \left| \frac{T_n}{1 + T_n \underline{m}(u_n)} + \frac{T}{1 + T \underline{m}(u_n)} \right|^2 \right)^{1/2} \\ & \leq M \left(E \left| \frac{1}{1 + T_n \underline{m}(u_n)} \right|^6 E \left| \frac{1}{1 + T \underline{m}(u_n)} \right|^6 E |T_n - T|^3 \right)^{1/3} \end{aligned}$$

converging to zero. Thus (7.35) is true, as claimed. We then conclude from the dominated convergence theorem that

$$\begin{aligned} & \int K'_r(z) \arg \left[1 - c \int \frac{t^2 (\underline{m}_n^0(z_n))^2}{(1 + t \underline{m}_n^0(z_n))^2} dH_n(t) \right] \\ & - \arg \left[1 - c \int \frac{t^2 (\underline{m}(u_n))^2}{(1 + t \underline{m}(u_n))^2} dH(t) \right] du \rightarrow 0. \end{aligned}$$

Moreover, by (7.7) we obtain

$$\int (K'_r(z) - K'_r(u)) \arg \left[1 - c \int \frac{t^2 (\underline{m}(u_n))^2}{(1 + t \underline{m}(u_n))^2} dH(t) \right] du \rightarrow 0.$$

By (7.21) and Theorem 1A in [10] (replacing $K(x)$ there by $K'(x)$) we see that

$$\int K'_r(u) \arg \left[1 - c \int \frac{t^2 (\underline{m}(u_n))^2}{(1 + t \underline{m}(u_n))^2} dH(t) \right] du \rightarrow 0.$$

Summarizing the above yields that (3.17) converges to zero.

Limits of (4.4) and (4.6). Repeating the argument leading to (7.16) yields (4.4) becomes

$$(7.36) \quad \frac{1}{h^2 \ln \frac{1}{h}} \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} K\left(\frac{u_1}{h}\right) K\left(\frac{u_2}{h}\right) \ln \left| \frac{\underline{m}(x_1 - u_1) - \overline{m}(x_2 - u_2)}{\underline{m}(x_1 - u_1) - \underline{m}(x_2 - u_2)} \right| du_1 du_2 + o(1).$$

The argument of (7.24) in Lemma 6 indeed also, together with (1.6), gives

$$(7.37) \quad \frac{1}{h^2} \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} K\left(\frac{u_1}{h}\right) K\left(\frac{u_2}{h}\right) g(x_1 - u_1, x_2 - u_2) du_1 du_2 - g(x_1, x_2) \rightarrow 0.$$

This ensures that (7.36) converges to zero when $x_1 \neq x_2$. When $x_1 = x_2 = x$, by (7.37) we have

$$\frac{1}{h^2 \ln \frac{1}{h}} \int_{x_1 - a_r}^{x_1 - a_l} \int_{x_2 - a_r - \varepsilon}^{x_2 - a_l + \varepsilon} K\left(\frac{u_1}{h}\right) K\left(\frac{u_2}{h}\right) \ln \left| \frac{\underline{m}(x_1 - u_1) - \overline{m}(x_2 - u_2)}{\underline{m}(x_1 - u_1) - \underline{m}(x_2 - u_2)} \right| du_1 du_2 \rightarrow 0.$$

Applying (7.37) and replacing $K'(x)$ in (7.5), (7.28), (7.29) and (7.30) by $K(x)$, we can prove that

$$-\frac{1}{h^2 \ln \frac{1}{h}} \int_{x_1-a_r}^{x_1-a_l} \int_{x_2-a_r-\varepsilon}^{x_2-a_l+\varepsilon} K\left(\frac{u_1}{h}\right) K\left(\frac{u_2}{h}\right) \ln \left| \underline{m}(x_1-u_1) - \underline{m}(x_2-u_2) \right| du_1 du_2 \rightarrow 1.$$

Checking on the argument of (3.17) and replacing $K'(x)$ there with $K(x)$, along with (4.3), we have

$$(4.6) \rightarrow 0.$$

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